Algebra Workbook

Higher Tier practice for Edexcel GCSE Mathematics (1MA1)

In-depth notes, formulae, worked examples and 160 practice questions covering the whole Algebra strand

Name: _____

Class / Tutor: _____

Start date: _____ Target grade: _____

All questions in this workbook are original practice questions written in the style of the specification. They are not past exam papers. Worked solutions and a mark scheme are in the separate Answer Booklet.

How to use this workbook

- Work through one section at a time. Read the **Concepts** pages first, then the **Formulae** and **Key facts**, then study every **Worked example**. Cover the solution and try each example yourself before attempting the practice questions.
- The **Common mistakes** box in each section lists the errors that cost the most marks. Read it again before you mark your work.
- Each practice question shows its marks and a grade level (Grade 3 to Grade 9). Do them in order: later questions in a section are harder.
- Show **all** working. In the real exam, method marks (M) are given for correct working even when the final answer is wrong.
- Calculator papers: use a scientific calculator. Where the answer is not exact, give 3 significant figures unless told otherwise.
- Mark your work with the Answer Booklet in a different colour, record your score below, and redo any question you lost marks on after three days.
- Suggested order: sections 1, 2, 4 and 6 first (the foundation), then 3, 5, 7 and 9, then the Higher-only topics 8, 10, 11 and 12.

Contents and progress tracker

Section	Topic	Page	Qs	Marks	My score	RAG
1	Expanding and factorising	5	15	41	/41	
2	Solving equations and inequalities	11	15	43	/43	
3	Quadratic equations and completing the square	17	14	45	/45	
4	Rearranging formulae	23	14	43	/43	
5	Sequences	28	14	45	/45	
6	Straight-line graphs	35	14	45	/45	
7	Quadratic and other graphs	42	13	46	/46	
8	Graph transformations	51	12	57	/57	
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10	Algebraic proof	68	14	45	/45	
11	Functions	75	12	46	/46	
12	Iteration	81	10	43	/43	

RAG: colour each topic Red, Amber or Green once you have checked your score. Green means 80% or more of the marks on grade 3 to 7 questions and at least half the marks on grade 8 to 9 questions.

Algebra formula quick reference

Edexcel Higher Tier does not give you most algebra formulae in the exam (the quadratic formula is NOT on the formulae page unless the paper says so). Learn all of these. Cover the right-hand column and test yourself.

Topic	Formula or rule
1. Difference of two squares	$a^2 - b^2 = (a - b)(a + b)$
1. Perfect squares	$(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$
1. Quadratic factorising	$x^2 + (p + q)x + pq = (x + p)(x + q)$
1. ac method	$ax^2 + bx + c$: find m, n with $mn = ac$ and $m + n = b$; split bx into mx + nx
1. Index laws	$a^m a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(a^m)^n = a^{mn}$
2. Inverse operations	+ ↔ −; × ↔ ÷; squaring ↔ square root
2. Inequality rule	multiply or divide by a negative: reverse the sign
2. Cross-multiplying	$a/b = c/d \rightarrow ad = bc$ ($b, d \neq 0$)
2. Quadratic inequality	if $(x - p)(x - q) < 0$ with $p < q$ then $p < x < q$; if > 0 then $x < p$ or $x > q$
3. Quadratic formula	$x = [-b \pm \sqrt{(b^2 - 4ac)}] / (2a)$
3. Discriminant	$b^2 - 4ac$: > 0 two roots; = 0 one root; < 0 no real roots
3. Completing the square (a = 1)	$x^2 + bx + c = (x + b/2)^2 + c - b^2/4$
3. Turning point	$y = a(x + p)^2 + q$ has turning point $(-p, q)$
3. Sum and product of roots	if the roots are α and β then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$
4. Standard rearrangements	$v = u + at \rightarrow t = (v - u)/a$; $A = \pi r^2 \rightarrow r = \sqrt{(A/\pi)}$; $v^2 = u^2 + 2as \rightarrow a = (v^2 - u^2)/(2s)$
4. Subject appears twice	collect terms → factorise → divide
4. Reciprocal formula	$1/f = 1/u + 1/v \rightarrow v = fu/(u - f)$
5. Arithmetic nth term	$T(n) = dn + (a - d)$, common difference d, first term a
5. Geometric nth term	$T(n) = a\, r^{n-1}$, common ratio r
5. Quadratic nth term	$T(n) = an^2 + bn + c$; a = (second difference) ÷ 2
5. Triangular numbers	$n(n + 1)/2$
5. Fibonacci type	$T(n + 2) = T(n + 1) + T(n)$
6. Line	$y = mx + c$; m = gradient, c = y-intercept
6. Gradient	$m = (y_2 - y_1) / (x_2 - x_1)$
6. Point-gradient form	$y - y_1 = m(x - x_1)$
6. Parallel / perpendicular	parallel: $m_1 = m_2$; perpendicular: $m_1 m_2 = -1$
6. Midpoint	$((x_1 + x_2)/2, (y_1 + y_2)/2)$
6. Distance	$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$
7. Quadratic graph	$y = ax^2 + bx + c$; roots where y = 0; y-intercept c; line of symmetry $x = -b/(2a)$
7. Turning point (from completed square)	$y = a(x + p)^2 + q$ has turning point $(-p, q)$
7. Circle, centre origin	$x^2 + y^2 = r^2$
7. Tangent to a circle	perpendicular to the radius: $m_{tangent} = -1 / m_{radius}$
7. Exponential	$y = k\, a^x$: passes through (0, k)
7. Reciprocal	$y = k/x$: asymptotes x = 0 and y = 0
8. Vertical translation	$y = f(x) + a$: translation (0, a)
8. Horizontal translation	$y = f(x + a)$: translation $(-a, 0)$
8. Reflections	$y = -f(x)$: in the x-axis; $y = f(-x)$: in the y-axis
8. Vertical stretch	$y = af(x)$: scale factor a parallel to the y-axis

8. Horizontal stretch	y = f(ax): scale factor 1/a parallel to the x-axis
8. Point (x, y) maps to	f(x) + a: (x, y + a); f(x + a): (x − a, y); −f(x): (x, −y); f(−x): (−x, y); af(x): (x, ay); f(ax): (x/a, y)
9. Elimination	same signs: SUBTRACT; different signs: ADD
9. Substitution	replace the letter in one equation with its equal from the other; use brackets
9. Line meets a curve	discriminant > 0: two points; = 0: tangent; < 0: no intersection
9. Circle	$x^2 + y^2 = r^2$; substitute the line, expand carefully
10. Even / odd	even: 2n; odd: 2n + 1; consecutive odds: 2n + 1, 2n + 3
10. Consecutive integers	n, n + 1, n + 2
10. Difference of two squares	$a^2 - b^2 = (a - b)(a + b)$
10. Always true	n(n + 1) is even; $(2n + 1)^2 = 4n(n + 1) + 1$
10. Perfect square	$(a - b)^2 \geq 0$ for all real a and b
11. Evaluate	f(a): replace x by a
11. Composite	fg(x) = f(g(x)): do g first
11. Inverse	y = f(x) → make x the subject → $f^{-1}(x)$
11. Inverse property	$f(f^{-1}(x)) = x = f^{-1}(f(x))$
11. Self-inverse	f(f(x)) = x
11. Graph of inverse	reflection of y = f(x) in the line y = x
12. Iteration	$x_{n+1} = g(x_n)$, starting at x_0
12. Change of sign	f(a) and f(b) opposite signs → a root between a and b
12. Limit	if $x_n \to L$ then L = g(L)
12. Staircase	vertical to the curve, horizontal to y = x, repeat

Laws of indices (check each one)

Rule	Example	Rule	Example
$a^m \times a^n = a^{m+n}$	$x^3 \times x^4 = x^7$	$a^m \div a^n = a^{m-n}$	$x^8 \div x^3 = x^5$
$(a^m)^n = a^{mn}$	$(x^2)^5 = x^{10}$	$a^0 = 1$	$7^0 = 1$
$a^{-n} = 1/a^n$	$2^{-3} = 1/8$	$a^{1/n}$ = n-th root of a	$27^{1/3} = 3$

1 Expanding and factorising

Grades 4 to 9 | 41 marks

Concepts explained

Why this topic matters

Expanding and factorising are opposite processes and they sit underneath almost every other algebra topic: solving quadratics, simplifying algebraic fractions, proof, completing the square and rearranging formulae all depend on them. If a student is slow or error-prone here, every later algebra topic suffers. Aim for accuracy first and speed second.

Language of algebra

- A **term** is a single piece of an expression, such as $5x^2$ or -7. Terms are separated by + and − signs.
- An **expression** has no equals sign ($3x + 2$). An **equation** can be solved ($3x + 2 = 11$). A **formula** links variables ($A = \pi r^2$). An **identity** is true for every value, written with ≡ (for example $(x + 1)^2 \equiv x^2 + 2x + 1$).
- **Like terms** have exactly the same letters and powers, so $4x^2$ and $-9x^2$ are like terms but $4x^2$ and $4x$ are not.

Laws of indices (you must know these)

To **multiply** powers of the same letter, add the indices: $a^m \times a^n = a^{m+n}$. To **divide**, subtract: $a^m \div a^n = a^{m-n}$. To take a **power of a power**, multiply: $(a^m)^n = a^{mn}$.

Also $a^0 = 1$, $a^{-n} = 1/a^n$, and $a^{1/n}$ is the n-th root of a. Remember the numbers in front (coefficients) are multiplied or divided normally: $3x^2 \times 4x^5 = 12x^7$.

Expanding single brackets

Multiply every term inside the bracket by the term outside, taking care with signs. A negative outside the bracket changes the sign of everything inside: $-2(x - 5) = -2x + 10$, not $-2x - 10$.

Expanding double brackets

Each term in the first bracket multiplies each term in the second. Use a **grid** or the FOIL order (First, Outer, Inner, Last), then collect like terms. Squaring a bracket means writing it twice: $(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$. The most common error in the whole of GCSE algebra is writing $(x + 3)^2 = x^2 + 9$.

For three brackets, expand any two first, simplify, then multiply by the third. A cubic result is always expected (x^3 term first).

Factorising: the four methods

1. Common factor. Take out the highest common factor of the numbers and of each letter. "Factorise fully" means nothing more can be taken out. Always check by expanding.

2. Quadratic $x^2 + bx + c$. Find two numbers that **multiply to c** and **add to b**. Then $x^2 + bx + c = (x + p)(x + q)$.

3. Difference of two squares. $a^2 - b^2 = (a - b)(a + b)$. It works only when there is a subtraction between two perfect squares. Look for it in disguise: $2x^2 - 50 = 2(x^2 - 25)$.

4. Quadratic $ax^2 + bx + c$ with $a \neq 1$. Multiply a × c. Find two numbers that multiply to ac and add to b. Split the middle term using those numbers, then factorise in pairs. Both pairs must leave the same bracket.

Simplifying algebraic fractions (Higher)

Treat an algebraic fraction like a numerical one: you can only cancel **factors** (things multiplied), never terms (things added). So factorise the top and the bottom completely, then cancel common brackets. You cannot cancel the x in $(x + 3)/x$.

To add or subtract, find a common denominator (usually the product of the two denominators), write both fractions over it, then combine the numerators. To multiply, multiply tops and bottoms (after factorising and cancelling). To divide, flip the second fraction and multiply.

Formulae and rules

Formula	Statement
Difference of two squares	$a^2 - b^2 = (a - b)(a + b)$
Perfect squares	$(a + b)^2 = a^2 + 2ab + b^2$; $(a - b)^2 = a^2 - 2ab + b^2$
Quadratic factorising	$x^2 + (p + q)x + pq = (x + p)(x + q)$
ac method	$ax^2 + bx + c$: find m, n with $mn = ac$ and $m + n = b$; split bx into $mx + nx$
Index laws	$a^m a^n = a^{m+n}$; $a^m \div a^n = a^{m-n}$; $(a^m)^n = a^{mn}$

Key facts

- Expand = remove brackets. Factorise = put brackets back. Always check a factorisation by expanding it again.
- Double bracket: $(a + b)(c + d) = ac + ad + bc + bd$. For a quadratic, the number of terms before collecting is 4 and after collecting is 3 (or 2 for a difference of squares).
- If the final term of $x^2 + bx + c$ is positive, the two brackets have the **same** sign. If it is negative, the brackets have **different** signs.
- To cancel in algebraic fractions you must factorise first. Only whole brackets or common factors cancel.

Worked examples

Example 1. Expand $3x(2x^2 - 5)$.

$3x \times 2x^2 = 6x^3$ and $3x \times (-5) = -15x$

$6x^3 - 15x$

Example 2. Expand and simplify $(x + 4)(x - 7)$.

$x \times x = x^2$; $x \times (-7) = -7x$; $4 \times x = 4x$; $4 \times (-7) = -28$

$x^2 - 7x + 4x - 28 =$ **$x^2 - 3x - 28$**

Example 3. Expand and simplify $(2x - 3)(3x + 5)$.

$6x^2 + 10x - 9x - 15$

= **$6x^2 + x - 15$**

Example 4. Expand and simplify $(x + 2)(x - 1)(x + 3)$.

$(x + 2)(x - 1) = x^2 + x - 2$

$(x^2 + x - 2)(x + 3) = x^3 + 3x^2 + x^2 + 3x - 2x - 6$

= **$x^3 + 4x^2 + x - 6$**

Example 5. Factorise fully $12x^2y - 18xy^2$.

HCF of 12 and 18 is 6; both terms contain x and y, so take out $6xy$

$12x^2y \div 6xy = 2x$ and $18xy^2 \div 6xy = 3y$

$6xy(2x - 3y)$

Example 6. Factorise $x^2 - 7x + 12$.

Need two numbers that multiply to +12 and add to −7: they are −3 and −4

$(x - 3)(x - 4)$

Check: $x^2 - 4x - 3x + 12 = x^2 - 7x + 12$ (check)

Example 7. Factorise $6x^2 + 7x - 5$.

$a \times c = 6 \times (-5) = -30$. Two numbers that multiply to -30 and add to $+7$: $+10$ and -3
Split the middle: $6x^2 + 10x - 3x - 5$
Factorise in pairs: $2x(3x + 5) - 1(3x + 5)$
$(2x - 1)(3x + 5)$

Example 8. Factorise $25x^2 - 49y^2$.

$25x^2 = (5x)^2$ and $49y^2 = (7y)^2$, with a subtraction between them
$(5x - 7y)(5x + 7y)$

Example 9. Simplify $(x^2 - 9) / (x^2 + 5x + 6)$.

Top: $x^2 - 9 = (x - 3)(x + 3)$. Bottom: $x^2 + 5x + 6 = (x + 2)(x + 3)$
Cancel the common bracket $(x + 3)$:
$(x - 3) / (x + 2)$

Example 10. Write $3/(x + 1) + 2/(x - 2)$ as a single fraction.

Common denominator $(x + 1)(x - 2)$
$[3(x - 2) + 2(x + 1)] / [(x + 1)(x - 2)] = (3x - 6 + 2x + 2) / [(x + 1)(x - 2)]$
$(5x - 4) / [(x + 1)(x - 2)]$

Common mistakes to avoid

- $(x + 3)^2 = x^2 + 9$ is wrong. The middle term $6x$ is missed. Always write the bracket twice.
- Forgetting that $-(x - 4) = -x + 4$ (both signs change).
- Factorising $12x^2y - 18xy^2$ as $3xy(4x - 6y)$. This is not fully factorised because $(4x - 6y)$ still has a common factor of 2.
- Cancelling terms such as the 2s in $(2x + 6)/2$ incorrectly as $x + 6$. Correct: $(2x + 6)/2 = x + 3$, because every term is divided.
- Using the difference of two squares on a sum: $x^2 + 16$ does not factorise.

Practice questions: 1. Expanding and factorising

1.1 Expand and simplify $3(2x - 5) - 2(x - 4)$. Grade 4

Answer ..... **[2 marks]**

1.2 Expand $4x(3x^2 - 2x + 5)$. Grade 4

Answer ..... **[2 marks]**

1.3 Expand and simplify $(x + 5)(x - 3)$. Grade 5

Answer ..... **[2 marks]**

1.4 Expand and simplify $(3x - 2)^2$. Grade 5

Answer ..... **[2 marks]**

1.5 Expand and simplify $(2x + 3)(5x - 4)$. Grade 5

Answer ..... **[2 marks]**

1.6 Factorise fully $8a^2b - 12ab^2$. Grade 5

Answer ..... **[2 marks]**

1.7 Factorise Grade 5

(a) $x^2 + 9x + 20$ **[2 marks]**

Answer .....

(b) $x^2 - 5x - 24$ **[2 marks]**

Answer .....

(c) $y^2 - 64$ **[1 mark]**

Answer .....

1.8 Factorise $3x^2 - 10x - 8$. Grade 6

Answer ..... **[3 marks]**

1.9 Factorise fully $2x^2 - 50$. Grade 6

Answer ..... **[3 marks]**

1.10 Expand and simplify $(x + 2)(x - 3)(x + 4)$. Grade 7

Answer ..... **[3 marks]**

1.11 Simplify fully $(x^2 + x - 12) / (x^2 - 9)$. Grade 7

Answer ..... **[3 marks]**

1.12 Write $3/(x + 2) + 4/(x - 1)$ as a single fraction. Give your answer in its simplest form. Grade 7

Answer ..... **[3 marks]**

1.13 Simplify fully $(2x^2 - 5x - 3) / (4x^2 - 1)$. Grade 8

Answer ..... **[3 marks]**

1.14 Simplify $(x^2 - 4)/(x + 3) \times (x^2 + 3x)/(x - 2)$. Grade 8

Answer ..... **[3 marks]**

1.15 Factorise fully $x^4 - 81$. Grade 9

Answer ..... **[3 marks]**

2 Solving equations and inequalities

Grades 3 to 8 | 43 marks

Concepts explained

The balance method

An equation is a balance. Whatever you do to one side you must do to the other. To solve, collect the unknown on one side and the numbers on the other by doing the **inverse operation**: undo + with −, × with ÷, and so on. Work in the reverse order of BIDMAS.

When x appears on both sides, first move the smaller x-term across (so the coefficient stays positive), then solve the resulting two-step equation. When there are brackets, expand first. When there are fractions, multiply every term by the common denominator to clear them.

Forming equations

Exam questions often hide the equation inside a context (angles, perimeters, ages, costs). Steps: (1) write down what is equal to what, for example "perimeter = 40" or "angles add to 180°", (2) substitute the expressions, (3) solve, (4) answer the actual question, which is often not x itself, and (5) check in the original situation.

Equations with x in the denominator (Higher)

Multiply both sides by the denominator, or multiply every term by the common denominator. For $(2x + 1)/(x − 3) = 4$, multiplying by $(x − 3)$ gives $2x + 1 = 4(x − 3)$, a linear equation. If two fractions equal each other, **cross-multiply**.

Inequalities

Solve an inequality exactly like an equation, with **one crucial difference**: when you multiply or divide both sides by a **negative** number, the inequality sign **reverses**. For example $−3x > 15$ gives $x < −5$.

On a number line, an **open circle** means the end value is not included (< or >), and a **filled circle** means it is included (≤ or ≥). A double inequality such as $5 ≤ 3x − 4 < 14$ is solved by doing the same operation to all three parts.

Integer solutions: list the whole numbers that satisfy the inequality, taking care at the ends (an open end excludes that integer).

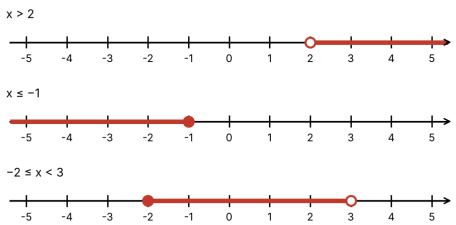
Quadratic inequalities (Higher)

Method: (1) solve the related equation to find the critical values, (2) sketch the quadratic (a U shape if the x^2 coefficient is positive), (3) read off the region. For $x^2 − x − 12 < 0$ the roots are −3 and 4 and the U is below the axis between them, so $−3 < x < 4$. For $x^2 − 2x − 8 ≥ 0$ the roots are −2 and 4 and the U is above the axis outside them, so $x ≤ −2$ or $x ≥ 4$. Never write the "outside" case as one combined inequality.

Inequalities on a graph (Higher)

A boundary line such as $y = 2x$ is drawn **solid** for ≤ or ≥ and **dashed** for < or >. Test a point (such as (0, 1) or (1, 0), not on the line) to decide which side to shade. The question may ask you to leave the region R **unshaded** or to shade the unwanted regions: read the instruction carefully.

Inequalities on a number line



Formulae and rules

Formula	Statement
Inverse operations	+ ↔ −; × ↔ ÷; squaring ↔ square root
Inequality rule	multiply or divide by a negative: reverse the sign
Cross-multiplying	a/b = c/d → ad = bc (b, d ≠ 0)
Quadratic inequality	if (x − p)(x − q) < 0 with p < q then p < x < q; if > 0 then x < p or x > q

Key facts

- Check every solution by substituting it into the original equation.
- Inequality symbols: < less than; > greater than; ≤ less than or equal; ≥ greater than or equal.
- Integer = whole number (…, −2, −1, 0, 1, 2, …). Do not forget negatives and 0 when the inequality allows them.
- Boundaries: ≤ and ≥ use a solid line or filled circle; < and > use a dashed line or open circle.

Worked examples

Example 1. Solve 5x − 7 = 2x + 14.

Subtract 2x: 3x − 7 = 14. Add 7: 3x = 21
x = 7

Example 2. Solve 3(2x − 1) = 4(x + 5).

6x − 3 = 4x + 20
2x = 23
x = 11.5

Example 3. Solve (x + 3)/2 + (x − 1)/3 = 5.

Multiply every term by 6: 3(x + 3) + 2(x − 1) = 30
3x + 9 + 2x − 2 = 30, so 5x + 7 = 30
5x = 23, so **x = 4.6**

Example 4. The sides of a rectangle are (x + 3) cm and (2x − 1) cm. The perimeter is 40 cm. Work out x and the area.

Perimeter: 2[(x + 3) + (2x − 1)] = 40
2(3x + 2) = 40, so 3x + 2 = 20 and x = **6**
Sides are 9 cm and 11 cm, so area = 9 × 11 = **99 cm^2**

Example 5. Solve (2x + 1)/(x − 3) = 4.

Multiply both sides by (x − 3): 2x + 1 = 4(x − 3)
2x + 1 = 4x − 12, so 13 = 2x
x = 6.5

Example 6. Solve −3x + 4 > 19.

−3x > 15
Divide by −3 and reverse the sign: **x < −5**

Example 7. Solve $5 \leq 3x - 4 < 14$ and list the integer solutions.

Add 4 to every part: $9 \leq 3x < 18$
Divide by 3: **$3 \leq x < 6$**
Integers: **3, 4, 5** (6 is not included)

Example 8. Solve $x^2 - x - 12 < 0$.

$x^2 - x - 12 = (x - 4)(x + 3) = 0$ at $x = 4$ and $x = -3$
The graph is a U shape, below the x-axis between the roots
$-3 < x < 4$

Example 9. Solve $x^2 - 2x - 8 \geq 0$.

$(x - 4)(x + 2) = 0$ at $x = 4$ and $x = -2$
U shape: on or above the axis outside the roots
$x \leq -2$ or $x \geq 4$

Common mistakes to avoid

- Not reversing the inequality sign when dividing by a negative number.
- Writing $x > 4$ or $x < -2$ as $-2 > x > 4$. The "or" case cannot be one chain.
- Only multiplying one term by the common denominator in an equation with fractions. Every term gets multiplied, including the whole numbers.
- Forgetting 0 or negative integers when listing integer solutions.
- Stopping at the value of x when the question asks for a length, an angle or an area.

Practice questions: 2. Solving equations and inequalities

2.1 Solve $4x + 7 = 31$. Grade 3

Answer ..... **[2 marks]**

2.2 Solve $7x - 3 = 4x + 15$. Grade 4

Answer ..... **[2 marks]**

2.3 Solve $3(x + 4) = 2(x - 5)$. Grade 5

Answer ..... **[3 marks]**

2.4 Solve $(3x - 1)/4 = 5$. Grade 5

Answer ..... **[2 marks]**

2.5 Solve $(x + 5)/3 + (x - 2)/2 = 7$. Grade 6

Answer ..... **[3 marks]**

2.6 Solve $6 - 2x \leq 14$. Grade 5

Answer ..... **[2 marks]**

2.7 Write down all the integer values of x that satisfy $-3 < 2x + 1 \leq 9$. Grade 6

Answer ..... **[3 marks]**

2.8 The sides of a triangle are x cm, (x + 4) cm and (2x − 2) cm. The perimeter is 38 cm. Work out the length of the longest side. Grade 6

Answer ..... **[3 marks]**

2.9 Solve $(2x + 1)/(x - 3) = 4$. Grade 7

Answer ..... **[3 marks]**

2.10 Solve $5/(x + 2) = 3/(x - 1)$. Grade 7

Answer ..... **[3 marks]**

2.11 Solve $3x - 1 < 2x + 5 < 4x + 1$. Grade 7

Answer ..... **[3 marks]**

2.12 Solve $(x - 2)/3 \geq (2x + 1)/5$. Grade 7

Answer ..... **[3 marks]**

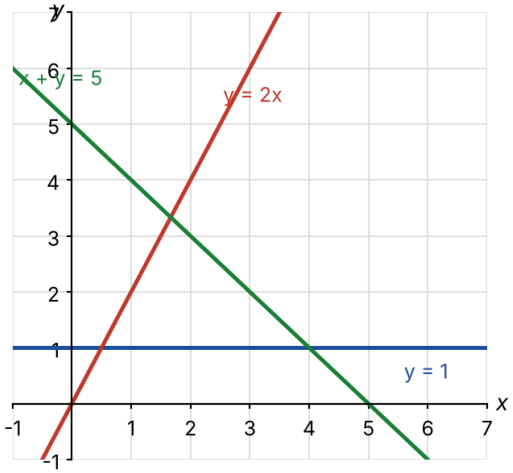
2.13 Solve $x^2 - 3x - 10 \geq 0$. Grade 7

Answer ..... **[3 marks]**

2.14 Solve $2x^2 + x - 6 < 0$. Grade 7

Answer ..... **[3 marks]**

2.15 The lines $y = 1$, $y = 2x$ and $x + y = 5$ are drawn on the grid. The region R satisfies all three inequalities $y \geq 1$, $y \leq 2x$ and $x + y \leq 5$. Grade 7



(a) On the grid, shade the region R and label it R. **[3 marks]**

Answer .....

(b) Write down the coordinates of every point with integer coordinates that lies inside R or on its boundary. **[2 marks]**

Answer .....

3 Quadratic equations and completing the square

Grades 5 to 9 | 45 marks

Concepts explained

What is a quadratic equation?

A quadratic equation can be written as **$ax^2 + bx + c = 0$**, where $a \neq 0$. It usually has up to two solutions (roots). The roots are the x-values where the graph $y = ax^2 + bx + c$ crosses the x-axis. Always rearrange so that one side is **zero** before solving.

Method 1: factorising

Factorise the quadratic, then use the fact that if two things multiply to give 0, one of them must be 0. From $(x - 7)(x + 2) = 0$ we get $x = 7$ or $x = -2$. Special cases: $x^2 - 4x = 0$ factorises as $x(x - 4) = 0$ (so $x = 0$ or 4: never divide by x), and $x^2 = 81$ gives $x = \pm 9$.

Method 2: the quadratic formula

When the quadratic does not factorise, use the formula. Write down a, b and c with their signs first. The value $b^2 - 4ac$ is called the **discriminant**. It tells you how many roots there are without solving.

- If $b^2 - 4ac > 0$ there are **two** different real roots (the graph crosses the x-axis twice).
- If $b^2 - 4ac = 0$ there is **one repeated** root (the graph touches the x-axis).
- If $b^2 - 4ac < 0$ there are **no real** roots (the graph does not reach the x-axis).

Common slips: forgetting that $-b$ is positive when b is negative, and putting the whole of "2a" under the square root.

Method 3: completing the square

Completing the square rewrites a quadratic as a squared bracket plus a number. For $x^2 + bx + c$: take half of b, so $x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c$. Example: $x^2 + 8x + 3 = (x + 4)^2 - 16 + 3 = (x + 4)^2 - 13$.

When $a \neq 1$, take a out of the x-terms first: $2x^2 - 12x + 5 = 2(x^2 - 6x) + 5 = 2[(x - 3)^2 - 9] + 5 = 2(x - 3)^2 - 13$.

Why it is useful: (1) it solves the equation exactly, leaving surds ($x = -4 \pm \sqrt{13}$); (2) it gives the turning point of the graph. For $y = a(x + p)^2 + q$ the turning point is at **$(-p, q)$** and the line of symmetry is $x = -p$. If $a > 0$ it is a minimum; if $a < 0$ it is a maximum.

Algebraic fractions leading to quadratics (Higher)

If an equation has x in the denominators, multiply every term by the common denominator to clear fractions. This often gives a quadratic. Reject any answer that makes a denominator zero.

Problem solving with quadratics

For area, Pythagoras or consecutive number problems, form an equation, rearrange to equal zero, solve, then reject solutions that do not make sense (a negative length, for example). Use the context to choose the valid root.

Formulae and rules

Formula	Statement
Quadratic formula	$x = [-b \pm \sqrt{(b^2 - 4ac)}] / (2a)$
Discriminant	$b^2 - 4ac$: > 0 two roots; = 0 one root; < 0 no real roots
Completing the square (a = 1)	$x^2 + bx + c = (x + b/2)^2 + c - b^2/4$
Turning point	$y = a(x + p)^2 + q$ has turning point $(-p, q)$

Sum and product of roots	if the roots are α and β then α + β = −b/a and αβ = c/a

Key facts

- Rearrange to $ax^2 + bx + c = 0$ first. Never solve (x + 3)(x − 2) = 14 by setting each bracket equal to 14.
- The quadratic formula is given on the Edexcel formulae sheet only in some papers. Learn it. It must be written with brackets round $b^2 - 4ac$ when you enter it into a calculator.
- Surd form: leave the answer as x = −4 ± √13 if the question says "exact" or "in surd form". Decimal form: give the stated number of decimal places or 3 s.f.
- Completing the square gives the turning point; factorising gives the roots; both are needed to sketch a quadratic.

Worked examples

Example 1. Solve $x^2 - 5x - 14 = 0$.

Numbers that multiply to −14 and add to −5: −7 and +2
(x − 7)(x + 2) = 0
x = 7 or x = −2

Example 2. Solve $3x^2 + x - 10 = 0$.

ac = −30; numbers −5 and +6
$3x^2 + 6x - 5x - 10 = 3x(x + 2) - 5(x + 2) = (3x - 5)(x + 2) = 0$
x = 5/3 or x = −2

Example 3. Solve $x^2 + 6x + 2 = 0$. Give your answers to 2 decimal places.

a = 1, b = 6, c = 2. $b^2 - 4ac = 36 - 8 = 28$
x = (−6 ± √28) / 2 = −3 ± √7
x = −0.35 or x = −5.65

Example 4. Solve $2x^2 - 7x + 1 = 0$. Give your answers to 2 decimal places.

a = 2, b = −7, c = 1. $b^2 - 4ac = 49 - 8 = 41$
x = (7 ± √41) / 4
x = 3.35 or x = 0.15

Example 5. (a) Write $x^2 + 8x + 3$ in the form $(x + a)^2 + b$. (b) Solve $x^2 + 8x + 3 = 0$ in surd form. (c) State the turning point.

(a) Half of 8 is 4: $(x + 4)^2 = x^2 + 8x + 16$, so we subtract 13: **$(x + 4)^2 - 13$**
(b) $(x + 4)^2 = 13$, so x + 4 = ±√13 and **x = −4 ± √13**
(c) Turning point (minimum) at **(−4, −13)**

Example 6. Write $2x^2 - 12x + 5$ in the form $a(x + b)^2 + c$.

$2(x^2 - 6x) + 5 = 2[(x - 3)^2 - 9] + 5$
$= 2(x - 3)^2 - 18 + 5$
$2(x - 3)^2 - 13$

Example 7. Solve 5/(x + 1) + 2/(x − 1) = 1. Give your answers to 2 decimal places.

Multiply by (x + 1)(x − 1): 5(x − 1) + 2(x + 1) = (x + 1)(x − 1)
$7x - 3 = x^2 - 1$, so $x^2 - 7x + 2 = 0$
x = (7 ± √41) / 2
x = 6.70 or x = 0.30

Example 8. Find the values of k for which $x^2 + kx + 9 = 0$ has equal roots.

Equal roots need $b^2 - 4ac = 0$
$k^2 - 4 \times 1 \times 9 = 0$, so $k^2 = 36$
k = 6 or k = −6

Common mistakes to avoid

- Not rearranging to zero first: from $(x + 3)(x - 2) = 14$ it is wrong to say $x + 3 = 14$.
- Losing the solution $x = 0$ by dividing both sides of $x^2 = 4x$ by x.
- In the formula, using b = 6 when the equation has −6x, or forgetting that $(-7)^2 = +49$.
- Completing the square: forgetting to subtract $(b/2)^2$, or not multiplying the subtracted number by a.
- Leaving answers as √ values in decimal questions, or rounding in the middle of a calculation.

Practice questions: 3. Quadratic equations and completing the square

3.1 Solve $x^2 + 7x + 10 = 0$. Grade 5

Answer ..... **[3 marks]**

3.2 Solve $x^2 - 4x = 0$. Grade 5

Answer ..... **[2 marks]**

3.3 Solve $x^2 - 81 = 0$. Grade 5

Answer ..... **[2 marks]**

3.4 Solve $2x^2 + 5x - 3 = 0$. Grade 6

Answer ..... **[3 marks]**

3.5 Solve $(x + 3)(x - 2) = 14$. Grade 6

Answer ..... **[3 marks]**

3.6 Solve $x^2 + 4x - 3 = 0$. Give your answers correct to 2 decimal places. Grade 6

Answer ..... **[3 marks]**

3.7 Solve $3x^2 - 8x - 2 = 0$. Give your answers correct to 2 decimal places. Grade 7

Answer ..... **[3 marks]**

3.8 Grade 6

(a) Write $x^2 - 10x + 7$ in the form $(x + a)^2 + b$. **[2 marks]**

Answer .....

(b) Hence solve $x^2 - 10x + 7 = 0$. Give your answers in the form $p \pm q\sqrt{2}$. **[2 marks]**

Answer .....

(c) Write down the coordinates of the turning point of the graph $y = x^2 - 10x + 7$. **[1 mark]**

Answer .....

3.9 Write $3x^2 + 12x + 5$ in the form $a(x + b)^2 + c$. Grade 7

Answer ..... **[3 marks]**

3.10 The rectangle has length $(x + 5)$ cm and width $(x - 2)$ cm. The area of the rectangle is 18 cm^2. Work out the length and the width. Grade 7

Answer ..... **[4 marks]**

3.11 Show that $2x^2 - 3x + 5 = 0$ has no real solutions. Grade 7

Answer ..... **[2 marks]**

3.12 The equation $x^2 + kx + 9 = 0$ has equal roots. Find the possible values of k. Grade 8

Answer ..... **[3 marks]**

3.13 Solve $3/(x + 2) + 2/(x - 1) = 1$. Give your answers correct to 2 decimal places. Grade 8

Answer ..... **[5 marks]**

3.14 The line $y = 2x + k$ is a tangent to the curve $y = x^2 + 4$. Work out the value of k. Grade 9

Answer ..... **[4 marks]**

4 Rearranging formulae

Grades 4 to 9 | 43 marks

Concepts explained

What does "make x the subject" mean?

The **subject** of a formula is the letter on its own on one side. In v = u + at the subject is v. Making t the subject means rearranging so that the formula reads t = …, with t appearing only once, on the left, and not on the right-hand side at all.

The method: undo, in reverse order

Think about what has been done to the letter you want, in BIDMAS order, then undo it with inverse operations in the **reverse** order. In $A = \pi r^2$, r is squared and then multiplied by π. Undo: divide by π, then square root. So $r = \sqrt{(A/\pi)}$.

Whatever you do to one side you must do to the whole of the other side. If the formula has a fraction, multiply both sides by the denominator first. If the formula has a square root, square both sides first.

When the subject appears twice (Higher)

If the letter you want appears in **more than one place**, you must collect all the terms containing it on one side and everything else on the other, then **factorise** to take the letter out as a common factor, then divide.

Example, $y = (3x + 1)/(x - 2)$: multiply by $(x - 2)$ to get $y(x - 2) = 3x + 1$; expand: $xy - 2y = 3x + 1$; collect x-terms on the left: $xy - 3x = 1 + 2y$; factorise: $x(y - 3) = 1 + 2y$; divide: $x = (1 + 2y)/(y - 3)$.

Squares and square roots

If the subject is squared, taking the square root gives **±**, unless the context rules one out (for example a length). From $y = 3(x + 2)^2$ we get $(x + 2)^2 = y/3$, so $x + 2 = \pm\sqrt{(y/3)}$, so $x = -2 \pm \sqrt{(y/3)}$.

Substituting into formulae

To evaluate a formula, write it down, replace each letter with its value in brackets (especially negative numbers), and use the calculator carefully. Do not round until the final answer.

Formulae and rules

Formula	Statement
Standard rearrangements	$v = u + at \rightarrow t = (v - u)/a$; $A = \pi r^2 \rightarrow r = \sqrt{(A/\pi)}$; $v^2 = u^2 + 2as \rightarrow a = (v^2 - u^2)/(2s)$
Subject appears twice	collect terms → factorise → divide
Reciprocal formula	$1/f = 1/u + 1/v \rightarrow v = fu/(u - f)$

Key facts

- Do the same to **both sides**. Always keep the equals sign in view.
- If the subject is inside a fraction, multiply to clear the fraction before anything else.
- If the subject appears twice or more, collect, factorise, divide. This is a grade 8 to 9 skill.
- Check by substituting simple numbers into the original and the rearranged formula.

Worked examples

Example 1. Make x the subject of $y = 5x - 3$.

$y + 3 = 5x$

$x = (y + 3)/5$

Example 2. Make r the subject of $A = \pi r^2$.

Divide by π: $A/\pi = r^2$
Square root: **$r = \sqrt{(A/\pi)}$** (r is a length, so use the positive root)

Example 3. Make a the subject of $v^2 = u^2 + 2as$.

$v^2 - u^2 = 2as$
$a = (v^2 - u^2)/(2s)$

Example 4. Make l the subject of $T = 2\pi\sqrt{(l/g)}$.

Divide by 2π: $T/(2\pi) = \sqrt{(l/g)}$
Square both sides: $T^2/(4\pi^2) = l/g$
Multiply by g: **$l = gT^2/(4\pi^2)$**

Example 5. Make x the subject of $ax + b = cx + d$.

Collect x-terms on the left: $ax - cx = d - b$
Factorise: $x(a - c) = d - b$
$x = (d - b)/(a - c)$

Example 6. Make x the subject of $y = (3x + 1)/(x - 2)$.

$y(x - 2) = 3x + 1$
$xy - 2y = 3x + 1$
$xy - 3x = 1 + 2y$
$x(y - 3) = 1 + 2y$
$x = (2y + 1)/(y - 3)$

Example 7. Make v the subject of $1/f = 1/u + 1/v$.

$1/v = 1/f - 1/u = (u - f)/(fu)$
Turn both sides upside down: **$v = fu/(u - f)$**

Example 8. Make x the subject of $y = 3(x + 2)^2$.

$y/3 = (x + 2)^2$
$x + 2 = \pm\sqrt{(y/3)}$
$x = -2 \pm \sqrt{(y/3)}$

Example 9. $s = ut + \frac{1}{2}at^2$. Work out s when $u = 6$, $a = 4$ and $t = 3$.

$s = 6 \times 3 + \frac{1}{2} \times 4 \times 3^2 = 18 + 18$
$s = 36$

Common mistakes to avoid

- Undoing in the wrong order. Think: what was done last? That is the first thing to undo.
- Subtracting a term from one side only, or forgetting that the "whole side" is divided: $x = y + 3/5$ instead of $(y + 3)/5$.
- Leaving the subject on both sides, for example $x = (2y + 3x)/5$.
- Taking the square root of each term separately: $\sqrt{(a^2 + b^2)}$ is not $a + b$.
- Splitting a fraction wrongly: $a/(b + c)$ is not $a/b + a/c$.

Practice questions: 4. Rearranging formulae

4.1 Make x the subject of $y = 4x + 7$. Grade 4

Answer ..... **[2 marks]**

4.2 Make t the subject of $v = u + at$. Grade 4

Answer ..... **[2 marks]**

4.3 Make b the subject of $A = \frac{1}{2}(a + b)h$. Grade 5

Answer ..... **[3 marks]**

4.4 Make y the subject of $3(x + y) = 2y + 5x$. Grade 5

Answer ..... **[3 marks]**

4.5 Make u the subject of $v^2 = u^2 + 2as$. Grade 6

Answer ..... **[2 marks]**

4.6 The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Make r the subject of the formula. Grade 6

Answer ..... **[3 marks]**

4.7 Make x the subject of $px + q = rx + s$. Grade 7

Answer ..... **[3 marks]**

4.8 Make l the subject of $T = 2\pi\sqrt{(l/g)}$. Grade 7

Answer ..... **[3 marks]**

4.9 Make x the subject of $y = 3(x - 2)^2$. Grade 7

Answer ..... **[3 marks]**

4.10 Make x the subject of $y = (2x + 3)/(x - 1)$. Grade 8

Answer ..... **[4 marks]**

4.11 $\mathbf{1/R = 1/R_1 + 1/R_2}$. Make R_2 the subject of this formula. Grade 8

Answer ..... **[3 marks]**

4.12 $x = (a + b)/(a - b)$. Make a the subject of the formula. Grade 8

Answer ..... **[4 marks]**

4.13 Show that if $y = 2x/(3 - x)$ then $x = 3y/(y + 2)$. Grade 8

Answer ..... **[3 marks]**

4.14 $q = \sqrt{[(p + 3)/(p - 1)]}$. Make p the subject of the formula. Grade 9

Answer ..... **[5 marks]**

5 Sequences

Grades 3 to 9 | 45 marks

Concepts explained

Terms, rules and notation

A **sequence** is an ordered list of numbers. Each number is a **term**. T(n) or u_n means the n-th term. A **term-to-term rule** tells you how to get from one term to the next ("add 3"). A **position-to-term rule** (the nth term) gives any term directly from its position n ("3n + 2").

Arithmetic (linear) sequences

The terms increase or decrease by the same amount each time, called the **common difference d**. The nth term is **dn + (a − d)**, where a is the first term. In other words: the coefficient of n is d, then add or subtract whatever is needed to match the first term.

Example: 5, 8, 11, 14. d = 3, so start with 3n: 3, 6, 9, 12. Compare with the sequence: add 2. nth term = **3n + 2**. Check n = 4: 3 × 4 + 2 = 14. (check).

Is a number in the sequence? Set the nth term equal to the number and solve. If n is a positive whole number, it is a term. If n is a fraction (for example 30.6) it is not.

Geometric sequences

Each term is the previous term multiplied by the same number, the **common ratio r**. 3, 6, 12, 24, … has r = 2. The nth term is $\mathbf{ar^{n-1}}$ where a is the first term. If 0 < r < 1 the terms decrease (80, 40, 20, 10, …). Geometric sequences with a negative ratio alternate in sign.

A geometric sequence is **not** linear: do not look for a constant difference.

Other sequences you should recognise

- Square numbers 1, 4, 9, 16 …: n^2. Cube numbers 1, 8, 27, 64 …: n^3.
- Triangular numbers 1, 3, 6, 10, 15 …: n(n + 1)/2. Powers of 2: 2, 4, 8, 16 …: 2^n.
- Fibonacci type: each term is the sum of the previous two. 1, 1, 2, 3, 5, 8 … The first two terms must be given.

Quadratic sequences (Higher)

A quadratic sequence has a constant **second difference**. Find the first differences, then the differences of those. The nth term is $an^2 + bn + c$ with **a = (second difference) ÷ 2**.

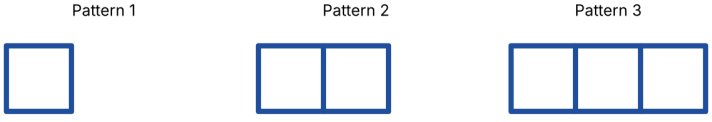
Method: (1) write the sequence, first differences and second differences; (2) a = second difference ÷ 2; (3) write down the terms of an^2 and **subtract** them from the original sequence; (4) the leftover terms form a linear sequence, whose nth term is bn + c; (5) put them together and check with n = 1 and n = 2.

Example: 5, 12, 23, 38, 57. First differences 7, 11, 15, 19; second differences 4, 4, 4 so a = 2. $2n^2$ gives 2, 8, 18, 32, 50; subtracting leaves 3, 4, 5, 6, 7, which is n + 2. nth term = $\mathbf{2n^2 + n + 2}$.

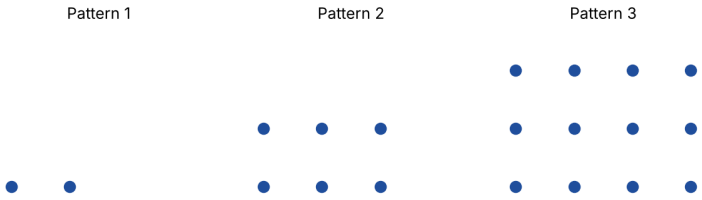
Pattern problems

For patterns made from sticks, tiles or dots, count the first few patterns, find the sequence, and find its nth term. Then explain the structure: in the matchstick pattern 3n + 1, each new square adds 3 sticks and there is 1 extra at the start. Examiners like "because" sentences that link the rule to the picture.

Matchstick pattern: 4, 7, 10, … (nth term 3n + 1)



Dot pattern: 2, 6, 12, … (nth term $n^2 + n$)



Formulae and rules

Formula	Statement
Arithmetic nth term	T(n) = dn + (a − d), common difference d, first term a
Geometric nth term	T(n) = a r^{n-1}, common ratio r
Quadratic nth term	T(n) = an^2 + bn + c; a = (second difference) ÷ 2
Triangular numbers	n(n + 1)/2
Fibonacci type	T(n + 2) = T(n + 1) + T(n)

Key facts

- First differences constant → linear. Second differences constant → quadratic. Constant ratio → geometric.
- The coefficient of n^2 is HALF the second difference. This is the most-tested fact.
- Always check the nth term with n = 1, 2 and 3.
- To test if a number is a term, solve nth term = number and check n is a positive integer.

Worked examples

Example 1. Find the nth term of the sequence 5, 8, 11, 14, …

Common difference 3, so 3n: 3, 6, 9, 12
Add 2 each time
nth term = 3n + 2

Example 2. Is 152 a term in the sequence 4, 9, 14, 19, …?

nth term = 5n − 1
5n − 1 = 152 gives 5n = 153 and n = 30.6
n is not a whole number, so **152 is not a term**

Example 3. The first term of a geometric sequence is 3 and the common ratio is 2. Write the first four terms, the nth term and the 8th term.

3, 6, 12, 24
nth term = $3 \times 2^{n-1}$
8th term = $3 \times 2^7 = 3 \times 128 =$ **384**

Example 4. A sequence starts 2, 5 and each term after the second is the sum of the previous two terms. Write the first six terms.

2, 5, 2 + 5 = 7, 5 + 7 = 12, 7 + 12 = 19, 12 + 19 = 31
2, 5, 7, 12, 19, 31

Example 5. Find the nth term of 5, 12, 23, 38, 57, …

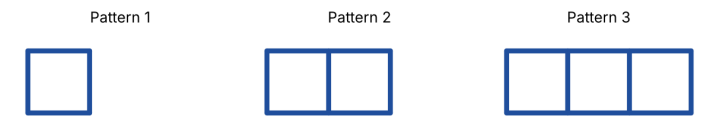
First differences: 7, 11, 15, 19. Second differences: 4, 4, 4
$a = 4 \div 2 = 2$. Subtract $2n^2$ (2, 8, 18, 32, 50): leaves 3, 4, 5, 6, 7
Linear part: $n + 2$
nth term = $2n^2 + n + 2$

Example 6. Find the nth term of 10, 7, 2, −5, −14, …

First differences: −3, −5, −7, −9. Second differences: −2
$a = -2 \div 2 = -1$. Subtract $-n^2$ (−1, −4, −9, −16, −25): leaves 11, 11, 11, 11, 11
nth term = $11 - n^2$

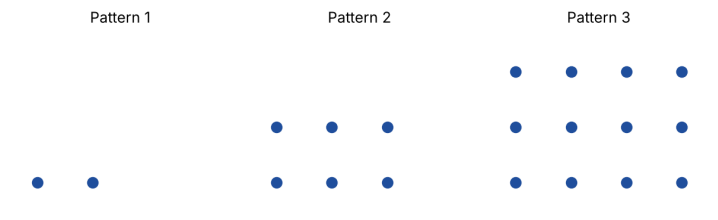
Example 7. The matchstick patterns are made of squares in a row. How many matchsticks are in pattern 20? Which pattern uses 100 matchsticks?

Sequence 4, 7, 10, … so nth term $3n + 1$ (each extra square adds 3 matchsticks)
Pattern 20: $3 \times 20 + 1 =$ **61**
$3n + 1 = 100$ gives $n = 33$, so **pattern 33**



Example 8. The dot patterns are shown. Find the nth term and the number of dots in pattern 10.

Dots: 2, 6, 12. Differences 4, 6; second difference 2, so $a = 1$
Pattern n has n rows of (n + 1) dots, so nth term = **$n(n + 1) = n^2 + n$**
Pattern 10: $10 \times 11 =$ **110 dots**



Common mistakes to avoid

- Writing the nth term as "n + 3" for 4, 7, 10: you must use the common difference as the coefficient of n.
- Using the second difference as the coefficient of n^2 instead of half of it.
- Not subtracting the n^2 terms before finding the linear part.
- Thinking 152 is a term because the arithmetic "nearly" works. n must be a positive integer.
- Describing a sequence by "add 3" when asked for the nth term. A term-to-term rule is not an nth term.

Practice questions: 5. Sequences

5.1 Write down the next two terms of the sequence 7, 3, −1, −5, … Grade 3

Answer ..... **[2 marks]**

5.2 Find the nth term of 7, 11, 15, 19, … Grade 4

Answer ..... **[2 marks]**

5.3 Find an expression for the nth term of 20, 17, 14, 11, … Grade 5

Answer ..... **[2 marks]**

5.4 The nth term of a sequence is 5n − 2. Grade 5

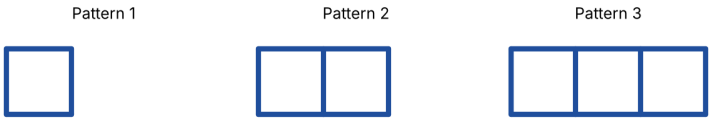
(a) Show that 148 is a term of the sequence. **[2 marks]**

Answer .....

(b) Is 200 a term of the sequence? Give a reason. **[2 marks]**

Answer .....

5.5 Here are the first three patterns of a sequence made from matchsticks. Grade 5



(a) How many matchsticks are in pattern 4? **[1 mark]**

Answer .....

(b) Find an expression for the number of matchsticks in pattern n. **[2 marks]**

Answer .....

(c) Which pattern uses exactly 91 matchsticks? **[2 marks]**

Answer .....

5.6 The first two terms of a sequence are 3 and 4. Each term after that is the sum of the two previous terms. Find the 6th term. Grade 5

Answer ..... **[2 marks]**

5.7 Here are the first four terms of a geometric sequence: 5, 10, 20, 40, … Grade 6

(a) Write down the next two terms. **[1 mark]**

Answer .....

(b) Work out the 10th term. **[2 marks]**

Answer .....

5.8 Find the nth term of the quadratic sequence 3, 9, 19, 33, 51, … Grade 7

Answer ..... **[3 marks]**

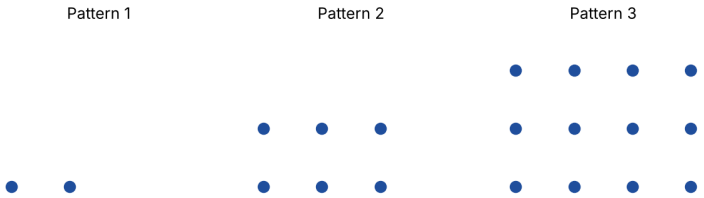
5.9 Find the nth term of the quadratic sequence 6, 11, 18, 27, 38, … Grade 7

Answer ..... **[3 marks]**

5.10 Find the nth term of 12, 9, 4, −3, −12, … Grade 7

Answer ..... **[3 marks]**

5.11 The diagrams show the first three patterns made from dots. Grade 7



(a) Work out the number of dots in pattern 10. **[2 marks]**

Answer .....

(b) Which pattern has 420 dots? **[2 marks]**

Answer .....

5.12 The nth term of a sequence is $n^2 + 3n + 1$. Grade 8

(a) Show that 55 is a term of the sequence. **[2 marks]**

Answer .....

(b) Show that 60 is not a term of the sequence. **[2 marks]**

Answer .....

5.13 The nth term of a sequence is $3n^2 - 2n$. Show that the difference between the (n + 1)th term and the nth term is 6n + 1. Grade 8

Answer ..... **[3 marks]**

5.14 The first three terms of an arithmetic sequence are $2x + 1$, $5x - 2$ and $6x + 3$. Grade 8

(a) Work out the value of x. **[3 marks]**

Answer .....

(b) Find an expression for the nth term. **[2 marks]**

Answer .....

6 Straight-line graphs

Grades 3 to 9 | 45 marks

Concepts explained

The equation y = mx + c

Every straight line can be written as **y = mx + c**. Here **m is the gradient** (steepness) and **c is the y-intercept** (where the line crosses the y-axis). Lines such as x = 3 are vertical and y = −2 are horizontal. Vertical lines have an undefined gradient.

Gradient

Gradient = change in y ÷ change in x = $(y^2 - y^1) / (x^2 - x^1)$. A line sloping up from left to right has a positive gradient; sloping down it has a negative gradient. On a graph, draw a gradient triangle as large as possible and read the sides carefully; the triangle must go in the direction of x increasing.

In a real-life graph the gradient is a **rate of change** (speed on a distance-time graph, cost per hour on a cost graph) and the intercept is the starting value or fixed charge.

Finding the equation of a line

From the graph: read the y-intercept for c, find the gradient for m. **From two points:** find m, then substitute one point into y = mx + c to find c, or use $y - y^1 = m(x - x^1)$. **From a point and a gradient:** use $y - y^1 = m(x - x^1)$ directly.

Equations like 3x + 4y = 12 must be rearranged to y = mx + c first: y = −¾x + 3, so the gradient is −¾ and the y-intercept is 3. To find the intercepts quickly, set x = 0 for the y-intercept and y = 0 for the x-intercept.

Parallel and perpendicular lines

Parallel lines have the same gradient. y = 3x − 1 and y = 3x + 7 are parallel.

Perpendicular lines: the gradients multiply to give −1: $m^1 \times m^2 = -1$. To find a perpendicular gradient, flip the fraction and change the sign: 2 becomes −½; −¾ becomes 4/3.

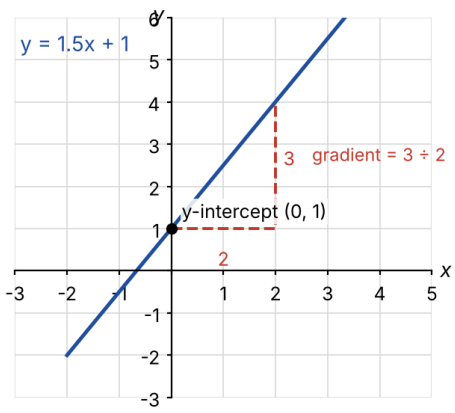
Midpoints, lengths and perpendicular bisectors

The **midpoint** of (x^1, y^1) and (x^2, y^2) is $((x^1 + x^2)/2, (y^1 + y^2)/2)$. The **length** of the line segment is $\sqrt{[(x^2 - x^1)^2 + (y^2 - y^1)^2]}$ (Pythagoras). The **perpendicular bisector** of AB passes through the midpoint of AB with gradient perpendicular to AB.

Where two lines meet

The coordinates of the intersection are found by solving the two equations simultaneously (see section 9). A point lies on a line if its coordinates satisfy the equation.

Gradient and y-intercept of a straight line



Formulae and rules

Formula	Statement
Line	$y = mx + c$; m = gradient, c = y-intercept
Gradient	$m = (y_2 - y_1) / (x_2 - x_1)$
Point-gradient form	$y - y_1 = m(x - x_1)$
Parallel / perpendicular	parallel: $m_1 = m_2$; perpendicular: $m_1 m_2 = -1$
Midpoint	$((x_1 + x_2)/2, (y_1 + y_2)/2)$
Distance	$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Key facts

- Vertical line: x = a. Horizontal line: y = b. The line y = x has gradient 1 and passes through the origin.
- Always rearrange to y = mx + c before reading off a gradient.
- Substitute coordinates to check a point is on a line: the equation must be a true statement.
- Draw a gradient triangle on the graph and label it. It earns method marks.

Worked examples

Example 1. Find the gradient and equation of the line through A(1, 2) and B(5, 10).

Gradient = (10 − 2)/(5 − 1) = 8/4 = 2
Use y − 2 = 2(x − 1), so y = 2x − 2 + 2
y = 2x

Example 2. Rearrange 3x + 4y = 12 into the form y = mx + c. Write down the gradient and the coordinates where the line crosses the axes.

4y = −3x + 12, so **y = −¾x + 3**
Gradient **−¾**, y-intercept **(0, 3)**
y = 0: 3x = 12, so x-intercept **(4, 0)**

Example 3. Find the equation of the line through (2, 5) and (6, −3).

m = (−3 − 5)/(6 − 2) = −8/4 = −2
y − 5 = −2(x − 2) so y = −2x + 4 + 5
y = −2x + 9
Check with (6, −3): −12 + 9 = −3 (check)

Example 4. Find the equation of the line parallel to y = 3x − 1 that passes through (2, 4).

Parallel, so the gradient is 3: y = 3x + c
4 = 3 × 2 + c so c = −2
y = 3x − 2

Example 5. Find the equation of the line perpendicular to 3y = x + 6 that passes through (1, 7).

3y = x + 6 gives y = ⅓x + 2, gradient ⅓
Perpendicular gradient: −3
7 = −3 × 1 + c so c = 10
y = −3x + 10

Example 6. A is (2, 1) and B is (8, 5). Find the equation of the perpendicular bisector of AB.

Midpoint M = ((2 + 8)/2, (1 + 5)/2) = (5, 3)
Gradient of AB = (5 − 1)/(8 − 2) = 4/6 = ⅔
Perpendicular gradient = −3/2
y − 3 = −1.5(x − 5), so y = −1.5x + 7.5 + 3
y = −1.5x + 10.5 (or 2y + 3x = 21)

Example 7. Does the point (7, 19) lie on the line y = 3x − 1?

y = 3 × 7 − 1 = 20
20 ≠ 19, so **no**, the point does not lie on the line

Example 8. A taxi charges a fixed £3 plus £2.50 per kilometre. Write down a formula for the cost y pounds for x km, and explain what the gradient and intercept mean.

y = 2.5x + 3
Gradient 2.5 = the cost per kilometre. y-intercept 3 = the fixed charge (cost for 0 km).

Common mistakes to avoid

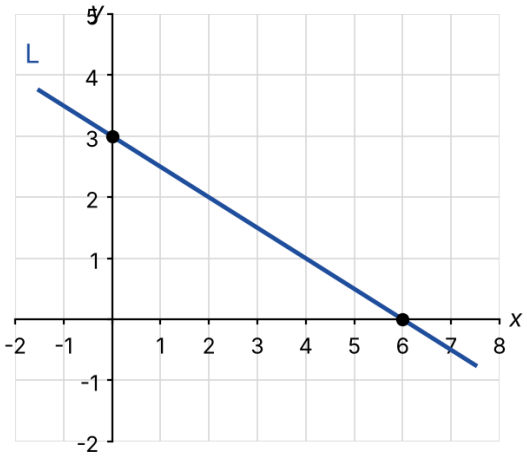
- Reading the gradient as "up over across" for a line going down without putting a minus sign.
- Using the gradient of the perpendicular as the negative reciprocal incorrectly: ½ becomes −2, not 2 or −½.
- Mixing x and y when finding gradient: it is (change in y)/(change in x).
- From 3x + 4y = 12, saying the gradient is 3, or −3. You must write y = … first.
- Giving the answer y = mx without c when the line does not go through the origin.

Practice questions: 6. Straight-line graphs

6.1 Work out the gradient of the straight line through (1, 3) and (5, 11). Grade 4

Answer ..... **[2 marks]**

6.2 Line L is shown on the grid. Find an equation of L. Grade 5



Answer ..... **[3 marks]**

6.3 The equation of a line is $2y = 6x - 8$. Write down the gradient and the y-intercept of the line. Grade 5

Answer ..... **[2 marks]**

6.4 Show that the point (3, 7) lies on the line $4x - y = 5$. Grade 5

Answer ..... **[2 marks]**

6.5 Find the coordinates of the point where the lines $y = 2x - 1$ and $3x + y = 14$ intersect. Grade 6

Answer ..... **[3 marks]**

6.6 Find an equation of the straight line through (2, 5) and (6, −3). Grade 6

Answer ..... **[3 marks]**

6.7 Find an equation of the line that is parallel to $y = 3x - 1$ and passes through (2, 4). Grade 6

Answer ..... **[2 marks]**

6.8 Line L_1 has the equation $3x + 2y = 12$. Work out the gradient of L_1 and the coordinates of the points where it crosses the axes. Grade 6

Answer ..... **[3 marks]**

6.9 A plumber charges a fixed call-out fee plus an hourly rate. A job lasting 1 hour costs £70. A job lasting 3 hours costs £130. Grade 6

(a) Work out the hourly rate. **[2 marks]**

Answer .....

(b) Write a formula for the cost £C of a job lasting h hours. **[2 marks]**

Answer .....

(c) Work out the cost of a job that takes 5.5 hours. **[1 mark]**

Answer .....

6.10 A is the point (−2, 3) and B is the point (4, −1). Grade 7

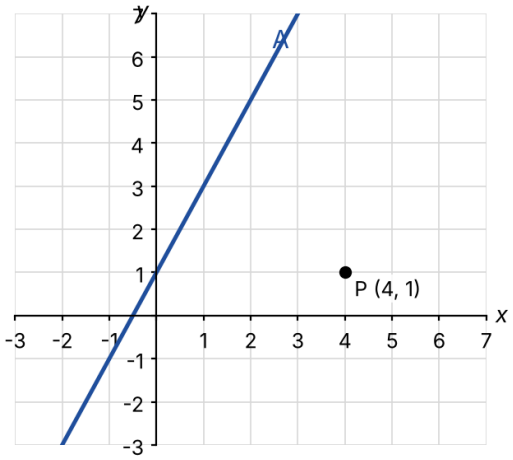
(a) Work out the coordinates of the midpoint of AB. **[2 marks]**

Answer .....

(b) Work out the length of AB. Give your answer in surd form. **[3 marks]**

Answer .....

6.11 The line A has the equation y = 2x + 1. P is the point (4, 1). Find an equation of the line through P that is perpendicular to A. Grade 7



Answer ..... **[3 marks]**

6.12 Find the equation of the line that passes through (0, −2) and is perpendicular to the line 2x + 3y = 6. Grade 8

Answer ..... **[3 marks]**

6.13 A is the point (2, 1) and B is the point (8, 5). Find an equation of the perpendicular bisector of AB. Grade 8

Answer ..... **[4 marks]**

6.14 The line L_1 has the equation $y = 3x + 2$. The line L_2 passes through A(0, 10) and is perpendicular to L_1. The lines meet at B. Work out the coordinates of B. Grade 9

Answer ..... **[5 marks]**

7 Quadratic and other graphs

Grades 4 to 9 | 46 marks

Concepts explained

Plotting a graph from a table of values

Substitute each x-value into the equation, work out y, plot the points and join them with a **smooth curve** (never with ruled straight segments). Use a pencil. If one point is clearly off the curve, recheck its calculation. For quadratics the table values are symmetrical about the turning point, which is a good check.

Features of a quadratic graph $y = ax^2 + bx + c$

- **Shape:** a "U" (a parabola) if $a > 0$, an upside-down "U" if $a < 0$.
- **Roots (x-intercepts):** where $y = 0$. Solve $ax^2 + bx + c = 0$.
- **y-intercept:** where $x = 0$. It is the number c.
- **Line of symmetry:** halfway between the roots, $x = -b/(2a)$.
- **Turning point:** on the line of symmetry. Find it by substituting the x-value back into the equation, or by completing the square: $y = a(x + p)^2 + q$ has turning point $(-p, q)$.

Sketching graphs

A **sketch** shows the shape and key points (axes intercepts, turning points) with their coordinates, but it is not plotted point by point. For a quadratic, find the roots, the y-intercept and the turning point. For a cubic such as $y = x(x - 2)(x + 1)$, find where it crosses the x-axis ($x = 0, 2, -1$) and use the shape: a positive x^3 cubic starts bottom left and ends top right.

Recognising the standard graphs

You should know the shape of: **linear** (a line), **quadratic** ($y = x^2$), **cubic** ($y = x^3$), **reciprocal** ($y = 1/x$, two branches with asymptotes $x = 0$ and $y = 0$), **exponential** ($y = a^x$ with $a > 1$: passes $(0, 1)$, always positive, gets steeper), and the **trigonometric** graphs $y = \sin x$, $y = \cos x$ (range -1 to 1, period 360°) and $y = \tan x$ (period 180°, asymptotes at 90° and 270°).

For $y = k \times a^x$, the y-intercept is k (when $x = 0$, $a^0 = 1$). Use a second point to find a.

Using graphs to solve equations

The solutions of $x^2 - 2x - 3 = 0$ are where the curve $y = x^2 - 2x - 3$ meets the line $y = 0$ (the x-axis). To solve $x^2 - 2x - 3 = 2$, draw the line $y = 2$ on the same axes and read the x-values where it meets the curve. In general, to solve an equation using a graph, find the line you need to add: it is the right-hand side after you arrange the equation to match the curve.

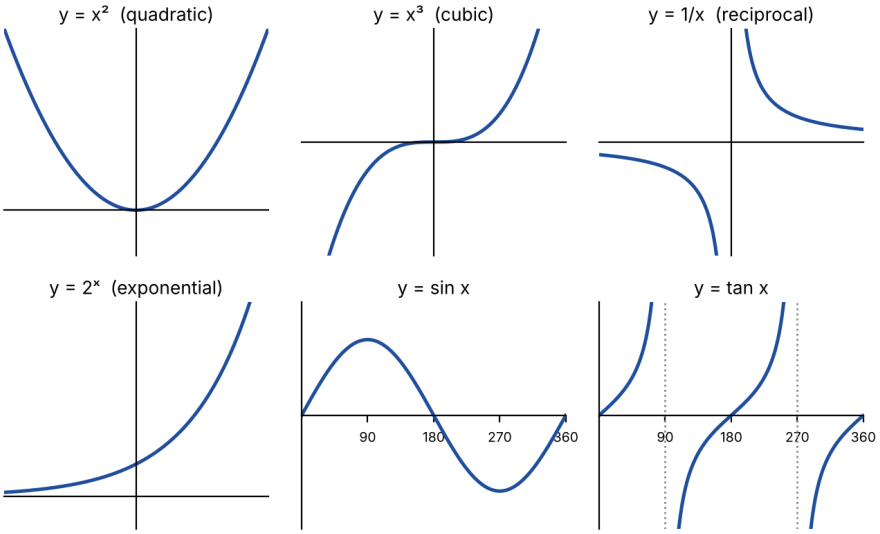
Circles and tangents (Higher)

The equation of a circle with centre at the origin and radius r is $\mathbf{x^2 + y^2 = r^2}$. A **tangent** to a circle touches it at exactly one point and is **perpendicular to the radius** at that point. To find the tangent at P: (1) gradient of the radius OP, (2) perpendicular gradient, (3) equation using $y - y^1 = m(x - x^1)$.

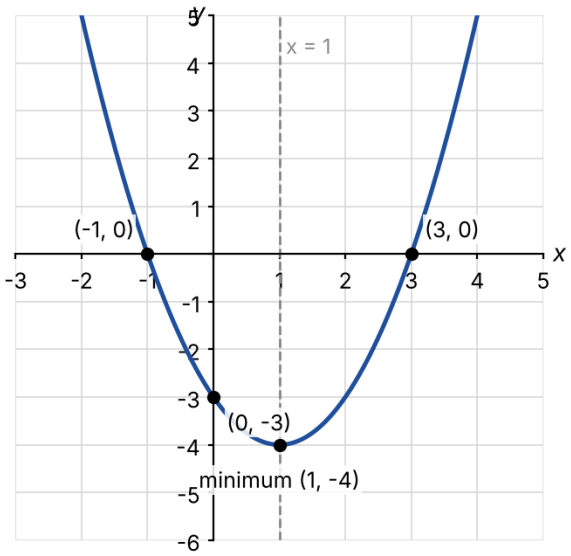
Gradient of a curve (Higher)

The gradient of a curve at a point is the gradient of the **tangent** to the curve at that point. You find it by drawing the tangent with a ruler and measuring its gradient. A **chord** joins two points on the curve; its gradient is an estimate of the gradient of the curve that gets better as the points get closer together. The gradient of a distance-time graph is speed; the gradient of a speed-time graph is acceleration.

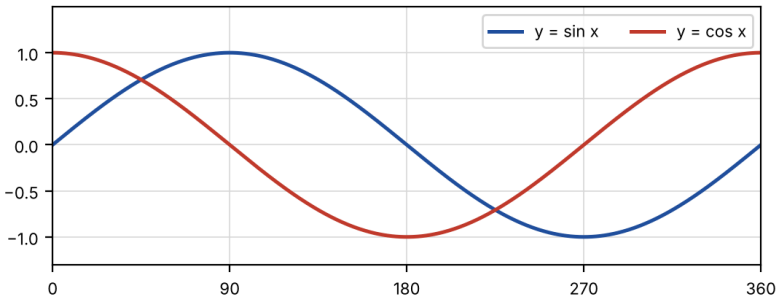
Standard graph shapes



Key features of $y = x^2 - 2x - 3$



y = sin x and y = cos x for 0° to 360°



Formulae and rules

Formula	Statement
Quadratic graph	$y = ax^2 + bx + c$; roots where $y = 0$; y-intercept c; line of symmetry $x = -b/(2a)$
Turning point (from completed square)	$y = a(x + p)^2 + q$ has turning point $(-p, q)$
Circle, centre origin	$x^2 + y^2 = r^2$

Tangent to a circle	perpendicular to the radius: $m_{tangent} = -1 / m_{radius}$
Exponential	$y = k\,a^x$: passes through (0, k)
Reciprocal	$y = k/x$: asymptotes x = 0 and y = 0

Key facts

- Plot points accurately and join with a smooth curve. Do not "flick" the ends or flatten the bottom of the U.
- To use a graph for an equation, write the equation as (curve) = (something simple), then draw that line.
- If a question says "use the graph", the answer must come from the graph, not from algebra. Show your working lines on the graph.
- The turning point of $x^2 - 6x + 5$ is halfway between the roots 1 and 5, that is at x = 3.

Worked examples

Example 1. Complete the table of values for $y = x^2 - 2x - 3$ for x from −2 to 4.

x = −2: 4 + 4 − 3 = 5. x = −1: 1 + 2 − 3 = 0. x = 0: −3. x = 1: 1 − 2 − 3 = −4.
x = 2: 4 − 4 − 3 = −3. x = 3: 9 − 6 − 3 = 0. x = 4: 16 − 8 − 3 = 5
y = 5, 0, −3, −4, −3, 0, 5. Note the symmetry about x = 1.

Example 2. For $y = x^2 - 6x + 5$, find the roots, the y-intercept and the turning point.

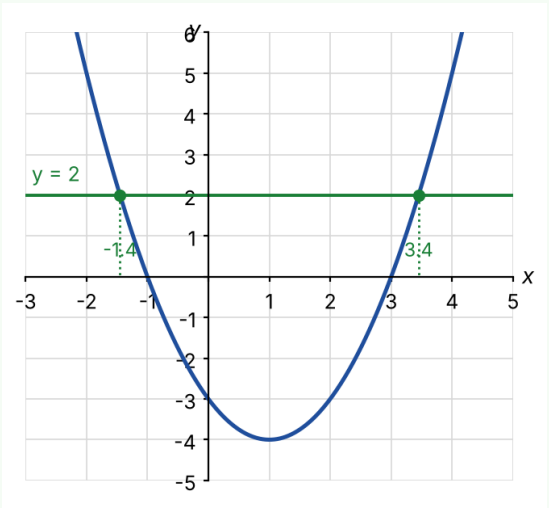
Roots: (x − 1)(x − 5) = 0 so **x = 1 and x = 5**
y-intercept: **(0, 5)**
Turning point: x = 3 (halfway between the roots), y = 9 − 18 + 5 = −4, so **(3, −4)**, a minimum

Example 3. Sketch y = x(x − 2)(x + 1).

Roots at x = −1, 0 and 2 (the curve crosses the x-axis at these points)
The x^3 coefficient is positive, so the curve starts low on the left and ends high on the right
Between −1 and 0 it is above the axis; between 0 and 2 it is below the axis
The y-intercept is (0, 0)

Example 4. The graph of $y = x^2 - 2x - 3$ is drawn. Use it to find estimates of the solutions of $x^2 - 2x - 3 = 2$.

Draw the line y = 2 on the same axes
Read the x-values where it crosses the curve:
x ≈ −1.4 and x ≈ 3.4

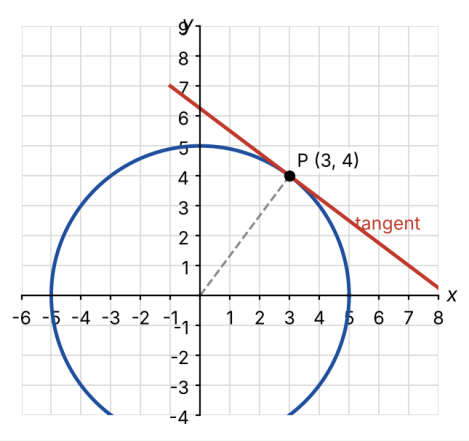


Example 5. The graph of $y = k \times a^x$ passes through (0, 5) and (2, 45). Find k and a.

At x = 0: k = 5
At x = 2: $5a^2 = 45$ so $a^2 = 9$ and a = 3
$y = 5 \times 3^x$

Example 6. Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point P(3, 4).

Gradient of OP = 4/3
Tangent is perpendicular so gradient = −3/4
$y - 4 = -\frac{3}{4}(x - 3)$ so $4y - 16 = -3x + 9$
$3x + 4y = 25$ (or $y = -\frac{3}{4}x + 25/4$)



Example 7. Use the graphs of sin x and cos x to solve sin x = 0.5 and cos x = −0.5 for 0° ≤ x ≤ 360°.

sin 30° = 0.5. By symmetry about 90° the other solution is 180° − 30° = 150°: **x = 30° or 150°**
cos 120° = −0.5. By symmetry about 180° the other solution is 360° − 120° = 240°: **x = 120° or 240°**

Example 8. Write $y = x^2 + 8x + 3$ in the form $y = (x + a)^2 + b$ and state the turning point and the line of symmetry.

$y = (x + 4)^2 - 13$
Turning point **(−4, −13)** (minimum); line of symmetry **x = −4**

Common mistakes to avoid

- Joining plotted points with straight lines or with a pointed bottom. A quadratic is smooth and symmetric.
- Mis-signing in the table: $(-2)^2 = +4$ but $-2^2 = -4$ on a calculator, so use brackets.
- Sketching a cubic with only one bend when it has three roots, or forgetting which way it goes at each end.
- Reading solutions off the wrong axis: roots are x-values.
- Taking the tangent gradient as the gradient of the radius. They are perpendicular.

Practice questions: 7. Quadratic and other graphs

7.1 Here is a table of values for $y = x^2 - 3x$. Grade 4

$x = -1, 0, 1, 2, 3, 4$

$y = 4, ____, -2, -2, ____, 4$

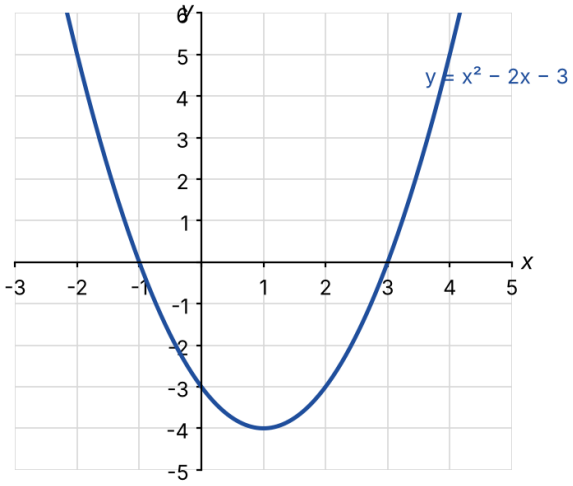
(a) Complete the table. **[2 marks]**

Answer .....

(b) Write down the equation of the line of symmetry of the graph. **[1 mark]**

Answer .....

7.2 The graph of $y = x^2 - 2x - 3$ is shown. Grade 5



(a) Write down the roots of $x^2 - 2x - 3 = 0$. **[1 mark]**

Answer .....

(b) Write down the coordinates of the turning point. **[1 mark]**

Answer .....

(c) Use the graph to estimate the solutions of $x^2 - 2x - 3 = 2$. Show your method on the graph. **[3 marks]**

Answer .....

7.3 Write down the equation of the graph of each type described. Choose from $y = x^2$, $y = x^3$, $y = 1/x$ and $y = 2^x$. Grade 6

(a) A U-shaped curve, symmetrical about the y-axis. **[1 mark]**

Answer .....

(b) A curve with two separate branches that never touches either axis. **[1 mark]**

Answer .....

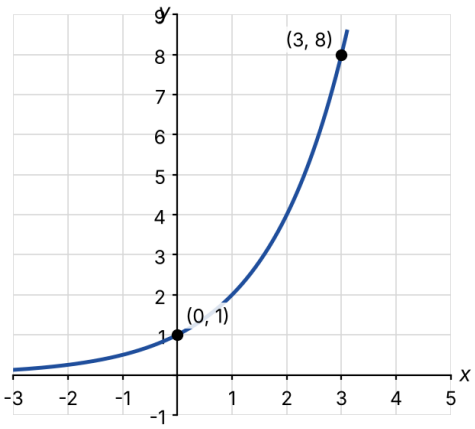
(c) A curve that passes through (0, 1) and (3, 8). **[1 mark]**

Answer .....

(d) A curve that passes through (−1, −1) and (1, 1) and has rotational symmetry about the origin. **[1 mark]**

Answer .....

7.4 The graph of $y = a^x$ passes through (0, 1) and (3, 8). Find the value of a. Grade 6



Answer ..... **[2 marks]**

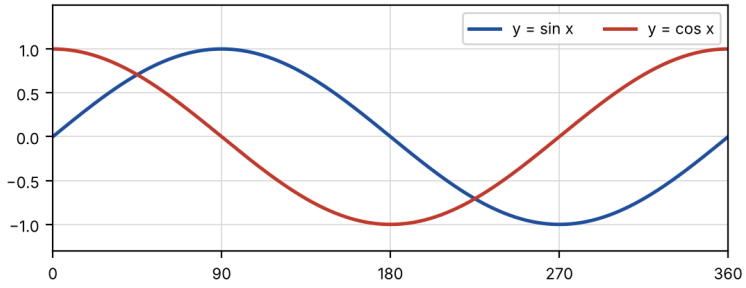
7.5 For the graph $y = x^2 - 6x + 5$, work out the coordinates of the points where the graph crosses the axes and the coordinates of the turning point. Then sketch the graph. Grade 6

Answer ..... **[4 marks]**

7.6 Sketch the graph of $y = x(x - 2)(x + 1)$. Mark the points where the graph crosses the axes. Grade 6

Answer ..... **[3 marks]**

7.7 Use the graph of $y = \sin x$ and $y = \cos x$ to find, for $0° \leq x \leq 360°$: Grade 6



(a) two solutions of $\sin x = 0.5$, given that $\sin 30° = 0.5$ **[2 marks]**

Answer .....

(b) two solutions of $\cos x = -0.5$, given that $\cos 120° = -0.5$ **[2 marks]**

Answer .....

7.8 Show that the point $(-3, 4)$ lies on the circle $x^2 + y^2 = 25$. Write down the equation of the circle with centre the origin and radius 7. Grade 6

Answer ..... **[3 marks]**

7.9 The curve $y = x^2 + bx + c$ has a turning point at (3, −5). Find the values of b and c. Grade 8

Answer ..... **[3 marks]**

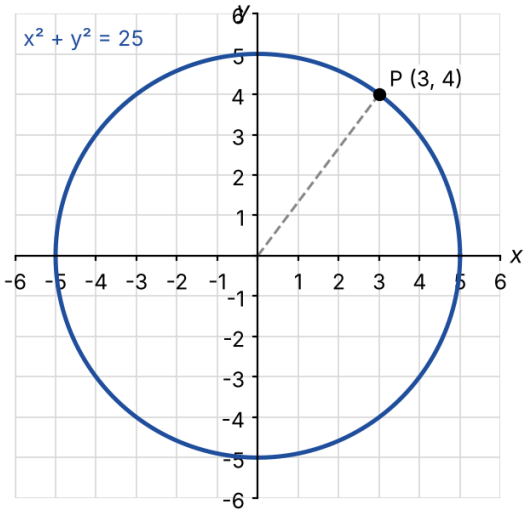
7.10 Show that the curve $y = x^2 - 6x + 11$ is always above the x-axis. Write down the coordinates of its turning point. Grade 8

Answer ..... **[3 marks]**

7.11 The graph of $y = k \times a^x$ passes through (0, 5) and (2, 45), and $a > 0$. Find k and a. Grade 8

Answer ..... **[3 marks]**

7.12 The circle $x^2 + y^2 = 25$ is shown. P is the point (3, 4). Find an equation of the tangent to the circle at P. Grade 8



Answer ..... **[4 marks]**

7.13 The curve with equation $y = x^2$ passes through A(1, 1), B(3, 9), C(1.9, 3.61) and D(2.1, 4.41). Grade 8

(a) Work out the gradient of the chord AB. **[1 mark]**

Answer .....

(b) Work out the gradient of the chord CD. **[2 marks]**

Answer .....

(c) Write down an estimate for the gradient of the curve at the point (2, 4). Explain why your estimate is reliable. **[2 marks]**

Answer .....

8 Graph transformations

Grades 5 to 9 | 57 marks

Concepts explained

The big idea

A transformation changes the graph y = f(x) into a new graph by changing the equation in a specific way. The key skill is to know how a change **inside** the brackets f( ) or **outside** them affects the picture. In the examples below we use the cubic $y = x^3 - 3x$, which has a maximum at (−1, 2), a minimum at (1, −2) and passes through (0, 0).

The rules

Changes outside the bracket affect y (vertical) and behave as you expect:

- y = f(x) + a moves the graph **up by a** (down if a is negative). Translation by the vector (0, a).
- y = −f(x) reflects the graph in the **x-axis** (y-values change sign).
- y = af(x) stretches the graph **vertically** by scale factor a (y-values multiplied by a, x-values unchanged).

Changes inside the bracket affect x (horizontal) and behave the opposite way to what you expect:

- y = f(x + a) moves the graph **left by a** (right if a is negative). Translation by (−a, 0). So y = f(x − 2) moves it right 2.
- y = f(−x) reflects the graph in the **y-axis** (x-values change sign).
- y = f(ax) stretches the graph **horizontally** by scale factor **1/a** (x-values divided by a, y-values unchanged). So y = f(2x) squeezes it towards the y-axis.

Why the inside works backwards

In y = f(x − 2) the output at x is what f gave at x − 2, so every feature of f has to be moved 2 further along to appear. A turning point at x = 1 on f(x) appears at x = 3 on f(x − 2). Substituting a test point is the best way to remember it.

Combined transformations

Apply the transformations to each key point in the order given by the rules for the function: **inside changes first (x-coordinate), outside changes after (y-coordinate)**. For y = f(x − 2) + 1 the point (−1, 2) goes to (1, 3) and the point (1, −2) goes to (3, −1). Describe transformations in words using the exact terms: translation, reflection in the x-axis, stretch with scale factor and the direction.

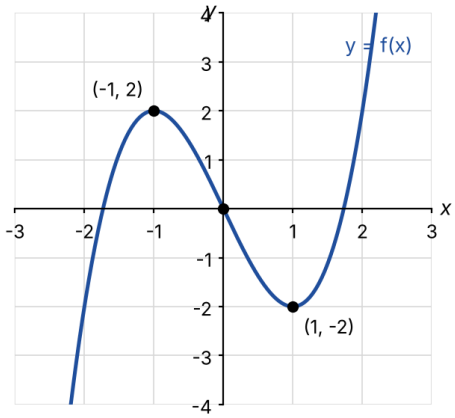
Quadratics and the completed square

The graph $y = x^2$ becomes $y = (x - p)^2 + q$ after a translation by the vector (p, q). The completed square form therefore tells you the transformation and the turning point (p, q) immediately. $y = -x^2 + 6$ is $y = x^2$ reflected in the x-axis and then moved up 6.

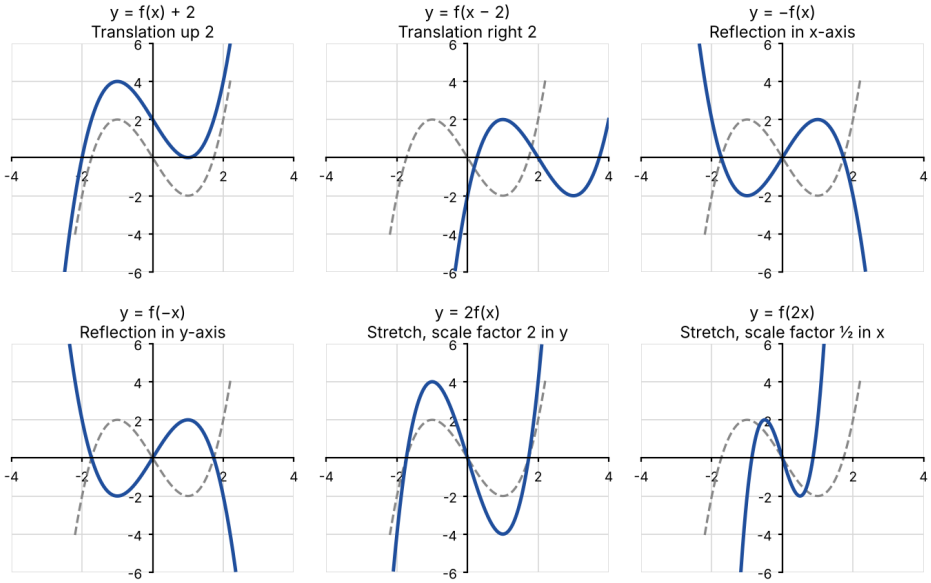
Trigonometric graphs

y = sin x + 1 has maximum 2 and minimum 0. y = 3 sin x has maximum 3 and minimum −3 (amplitude 3). y = sin 2x completes a cycle in 180° (period halves). y = sin (x + 30°) is shifted 30° to the left.

y = f(x)



Six transformations of y = f(x) (dashed: original)



Formulae and rules

Formula	Statement
Vertical translation	y = f(x) + a: translation (0, a)
Horizontal translation	y = f(x + a): translation (−a, 0)
Reflections	y = −f(x): in the x-axis; y = f(−x): in the y-axis
Vertical stretch	y = af(x): scale factor a parallel to the y-axis
Horizontal stretch	y = f(ax): scale factor 1/a parallel to the x-axis
Point (x, y) maps to	f(x) + a: (x, y + a); f(x + a): (x − a, y); −f(x): (x, −y); f(−x): (−x, y); af(x): (x, ay); f(ax): (x/a, y)

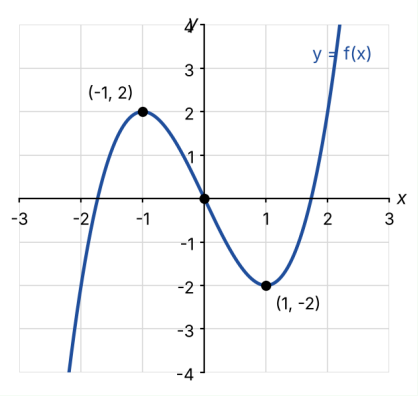
Key facts

- Inside the bracket: x-direction and opposite. Outside the bracket: y-direction and as expected.
- Apply the rule to the coordinates of the key points (turning points, intercepts) and then sketch.
- Always write the new coordinates for each key point on your sketch.
- Describe a transformation using the technical word and the details: "translation by vector (3, 0)", not "moved over".

Worked examples

Example 1. The graph of y = f(x) has a maximum at (−1, 2) and a minimum at (1, −2). Write down the coordinates of the turning points of y = f(x) + 3.

Add 3 to each y-coordinate: **(−1, 5) and (1, 1)**



Example 2. Write down the turning points of y = f(x − 2).

Move 2 to the right (add 2 to each x-coordinate): **(1, 2) and (3, −2)**

Example 3. Write down the turning points of y = −f(x).

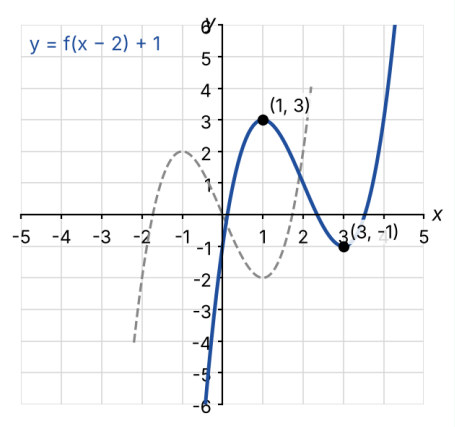
Change the sign of each y-coordinate: **(−1, −2) and (1, 2)**. The maximum becomes a minimum.

Example 4. Write down the turning points of y = f(2x) and y = 2f(x).

f(2x): divide each x-coordinate by 2: **(−0.5, 2) and (0.5, −2)**
2f(x): multiply each y-coordinate by 2: **(−1, 4) and (1, −4)**

Example 5. Sketch y = f(x − 2) + 1 and mark the turning points.

(−1, 2) → (−1 + 2, 2 + 1) = (1, 3); (1, −2) → (3, −1)
The shape is unchanged and the curve is 2 right and 1 up.



Example 6. Describe fully the transformation that maps y = f(x) onto (a) y = f(x + 4), (b) y = f(x) − 5, (c) y = f(x/3).

(a) Translation by vector (−4, 0) (4 units left)
(b) Translation by vector (0, −5) (5 units down)
(c) Stretch parallel to the x-axis with scale factor 3

Example 7. The curve $y = x^2$ is translated by the vector (3, 2). Write down the equation of the new curve and its turning point.

Replace x by (x − 3) and add 2: **$y = (x - 3)^2 + 2$**
Turning point **(3, 2)**

Example 8. State the maximum and minimum values of $y = 3 \sin x + 1$.

sin x lies between −1 and 1, so 3 sin x lies between −3 and 3
Adding 1: **maximum 4, minimum −2**

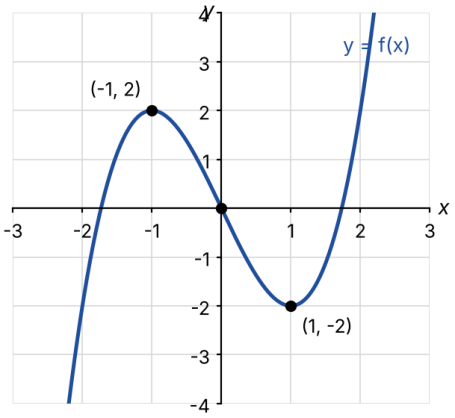
Common mistakes to avoid

- Moving f(x + 2) to the right. Remember: inside the bracket does the opposite.
- Using the vector (2, 0) for f(x + 2). It is (−2, 0).
- For f(2x) multiplying the x-coordinates by 2. They are divided by 2.
- Mixing up reflection in the x-axis (−f(x)) with reflection in the y-axis (f(−x)).
- Describing a transformation as "shifted" or "moved" without a vector or scale factor.

Practice questions: 8. Graph transformations

8.1 The graph of y = f(x) is shown. It has a maximum at (−1, 2) and a minimum at (1, −2). Write down the coordinates of the turning points of the graph of

Grade 6



(a) y = f(x) + 3 **[2 marks]**

Answer .....

(b) y = f(x + 2) **[2 marks]**

Answer .....

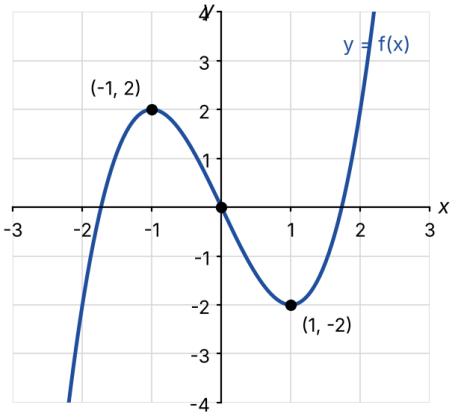
(c) y = −f(x) **[2 marks]**

Answer .....

(d) y = f(−x) **[2 marks]**

Answer .....

8.2 The graph of y = f(x) is shown. Write down the coordinates of the turning points of



(a) y = 2f(x) **[2 marks]**

Answer .....

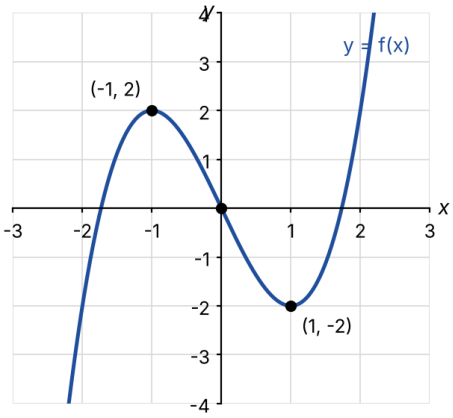
(b) y = f(2x) **[2 marks]**

Answer .....

(c) y = f(x/2) **[2 marks]**

Answer .....

8.3 On the grid the graph of y = f(x) is drawn. Sketch the graph of y = f(x − 2) + 1 and write the coordinates of its turning points. Grade 7



Answer ..... **[3 marks]**

8.4 Describe fully the single transformation that maps the graph of y = f(x) onto the graph of Grade 6

(a) y = f(x) + 7 **[2 marks]**

Answer .....

(b) y = f(x − 5) **[2 marks]**

Answer .....

(c) y = −f(x) **[2 marks]**

Answer .....

(d) y = f(3x) **[2 marks]**

Answer .....

8.5 The graph of $y = x^2$ is transformed. Grade 7

(a) Write down the equation of the graph after a translation by the vector (3, 2). **[2 marks]**

Answer .....

(b) Write down the turning point of the graph $y = (x + 4)^2 - 5$. **[1 mark]**

Answer .....

(c) The graph of $y = x^2$ is reflected in the x-axis and then translated up by 6. Write down its equation. **[2 marks]**

Answer .....

8.6 The graph of $y = \sin x$ is transformed. For each new graph, write down the maximum value, and for (c) the period. Grade 7

(a) $y = 3 \sin x$ **[1 mark]**

Answer .....

(b) $y = \sin x + 2$ **[1 mark]**

Answer .....

(c) $y = \sin 2x$. Write down the period. **[1 mark]**

Answer .....

(d) $y = \cos x + 2$. Write down the minimum value. **[1 mark]**

Answer .....

8.7 The graph of $y = f(x)$ has a minimum at $(2, -3)$. Write down the coordinates of the minimum of Grade 7

(a) $y = f(x) + 5$ **[1 mark]**

Answer .....

(b) $y = f(x - 1)$ **[1 mark]**

Answer .....

(c) $y = 3f(x)$ **[1 mark]**

Answer .....

(d) $y = f(2x)$ **[1 mark]**

Answer .....

8.8 The graph of $y = f(x)$ has a maximum point at (4, 6). Write down the coordinates of the turning point of the graph of Grade 8

(a) $y = 2f(x) + 1$ **[2 marks]**

Answer .....

(b) $y = f(x/2)$ **[2 marks]**

Answer .....

(c) $y = -f(x - 3)$. State whether it is a maximum or a minimum. **[3 marks]**

Answer .....

8.9 The curve $y = x^2 - 6x + 11$ is translated by the vector $(-2, 3)$. Find the equation of the new curve in the form $y = x^2 + bx + c$. Grade 8

Answer ..... **[3 marks]**

8.10 The graph of y = f(x) passes through the point (2, 5). Work out the coordinates of the point on y = 2f(x − 3) + 1 that corresponds to it. Grade 8

Answer ..... **[3 marks]**

8.11 The graph of $y = x^2 - 4x + 7$ is reflected in the y-axis. Find the equation of the new graph. Grade 8

Answer ..... **[3 marks]**

8.12 The curve with equation y = f(x) has a turning point at (a, b). The curve with equation y = f(3x) − 4 has a turning point at (6, 1). Work out a and b. Grade 9

Answer ..... **[3 marks]**

9 Simultaneous equations

Grades 4 to 9 | 50 marks

Concepts explained

What are simultaneous equations?

Two (or more) equations with the same unknowns, which must both be true at the same time. The solution is the pair (x, y) that satisfies both. Graphically it is the point where the two graphs meet. Two lines can meet at one point, never (parallel), or everywhere (the same line).

Method 1: elimination

(1) Label the equations (1) and (2). (2) Multiply one or both equations so that the coefficients of one letter are equal in size. (3) **Same signs: subtract. Different signs: add.** (4) Solve for the remaining letter. (5) Substitute back into an original equation to find the other. (6) Check in the **other** original equation.

Example: $3x + 2y = 12$ (1), $5x - 2y = 4$ (2). The y-terms have opposite signs and equal size, so add: $8x = 16$, $x = 2$. Then $6 + 2y = 12$ so $y = 3$.

Method 2: substitution

Use when one equation already has a letter on its own ($y = \ldots$) or is easy to rearrange. Substitute the expression into the other equation, use brackets, solve, then substitute back. Substitution is **the** method for linear-quadratic and linear-circle pairs.

Forming simultaneous equations from a problem

Define the unknowns (let a = the price of an adult ticket, c = the price of a child ticket), write one equation for each piece of information, solve, then give the answer with units in the context of the question.

Linear and quadratic pairs (Higher)

Substitute the linear equation into the quadratic to get a quadratic in one variable, solve it, then substitute each answer back into the **linear** equation to find y. There are normally **two** solutions (x, y) pairs. Write them as paired coordinates. Using the quadratic to find y can give wrong pairs.

The discriminant of the resulting quadratic tells you the geometry: $b^2 - 4ac > 0$ two intersections (a secant), $= 0$ one point (the line is a **tangent**), < 0 none.

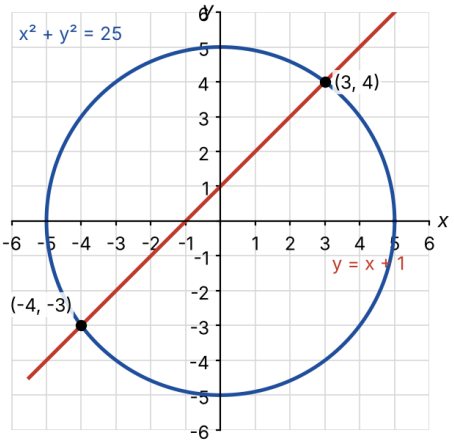
Linear and circle pairs (Higher)

For $x^2 + y^2 = 25$ and $y = x + 1$, substitute y: $x^2 + (x + 1)^2 = 25$. Expand $(x + 1)^2$ carefully (three terms). Tidy to a quadratic and solve. Check that both points satisfy both equations.

Graphical solutions

Draw both graphs accurately on the same axes and read off the coordinates of the intersection. Graphical answers are estimates: give them to a sensible accuracy.

The circle $x^2 + y^2 = 25$ and the line $y = x + 1$ meet at (3, 4) and (−4, −3)



Formulae and rules

Formula	Statement
Elimination	same signs: SUBTRACT; different signs: ADD
Substitution	replace the letter in one equation with its equal from the other; use brackets
Line meets a curve	discriminant > 0: two points; = 0: tangent; < 0: no intersection
Circle	$x^2 + y^2 = r^2$; substitute the line, expand carefully

Key facts

- Always check your answer in the equation you did not use for the final substitution.
- Brackets matter: substituting $y = x + 1$ into $x^2 + y^2$ gives $x^2 + (x + 1)^2$, not $x^2 + x + 1^2$.
- Give a solution as a pair: $x = 3$, $y = 4$ (or (3, 4)), and list both pairs for non-linear systems.
- In a context question, always answer the question asked, with units.

Worked examples

Example 1. Solve $3x + 2y = 12$ and $5x - 2y = 4$.

Add (the y-terms have opposite signs): $8x = 16$ so **$x = 2$**
Substitute into the first equation: $6 + 2y = 12$ so **$y = 3$**
Check in the second: $10 - 6 = 4$ (check)

Example 2. Solve $2x + 3y = 7$ and $3x + 5y = 11$.

Multiply (1) by 3: $6x + 9y = 21$. Multiply (2) by 2: $6x + 10y = 22$
Same signs, so subtract: $y = 1$
Substitute: $2x + 3 = 7$ so $x = 2$
$x = 2$, $y = 1$. Check: $6 + 5 = 11$ (check)

Example 3. Solve $y = 2x - 1$ and $3x + y = 14$.

Substitute: $3x + (2x - 1) = 14$
$5x = 15$ so $x = 3$, and $y = 2 \times 3 - 1 = 5$
$x = 3$, $y = 5$

Example 4. Three adult tickets and two child tickets cost £34. Two adult tickets and five child tickets cost £41. Work out the cost of one adult ticket and one child ticket.

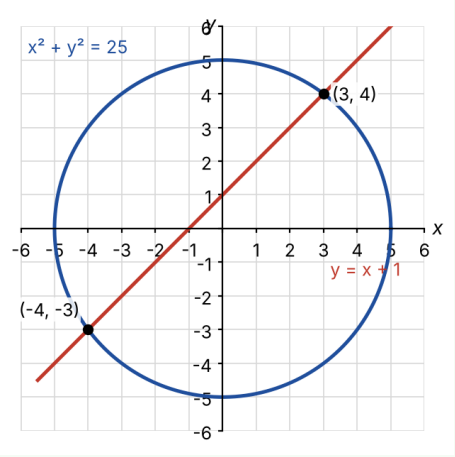
Let a = adult price, c = child price: $3a + 2c = 34$ (1), $2a + 5c = 41$ (2)
(1) × 5: $15a + 10c = 170$. (2) × 2: $4a + 10c = 82$
Subtract: $11a = 88$ so $a = 8$. Then $24 + 2c = 34$ so $c = 5$
Adult £8, child £5. Check (2): $16 + 25 = 41$ (check)

Example 5. Solve $y = x + 2$ and $y = x^2 - 4$.

$x + 2 = x^2 - 4$ so $x^2 - x - 6 = 0$
$(x - 3)(x + 2) = 0$ so $x = 3$ or $x = -2$
Substitute into $y = x + 2$: $y = 5$ or $y = 0$
(3, 5) and (−2, 0)

Example 6. Solve $x^2 + y^2 = 25$ and $y = x + 1$.

$x^2 + (x + 1)^2 = 25$
$x^2 + x^2 + 2x + 1 = 25$ so $2x^2 + 2x - 24 = 0$, $x^2 + x - 12 = 0$
$(x + 4)(x - 3) = 0$ so $x = -4$ or $x = 3$
$y = x + 1$: **(3, 4) and (−4, −3)**



Example 7. Show that the line $y = x + 5$ is a tangent to the curve $y = x^2 - x + 6$.

$x^2 - x + 6 = x + 5$ so $x^2 - 2x + 1 = 0$
$(x - 1)^2 = 0$ has one repeated root $x = 1$, so the line touches the curve at one point
The line is a tangent, touching at (1, 6).

Common mistakes to avoid

- Adding when you should subtract (or the reverse). Same signs subtract; different signs add.
- Multiplying only one term of an equation when scaling. Multiply every term, including the right-hand side.
- Finding x only and stopping. Both unknowns are needed.
- For linear-quadratic pairs, giving two x-values and two y-values unpaired. Pair them using the linear equation.
- Forgetting the middle term when expanding $(x + 1)^2$ in circle problems.

Practice questions: 9. Simultaneous equations

9.1 Solve $x + y = 10$ and $x - y = 4$. Grade 5

Answer ..... **[3 marks]**

9.2 Solve $3x + 2y = 12$ and $5x - 2y = 4$. Grade 5

Answer ..... **[3 marks]**

9.3 Solve $y = 3x - 2$ and $2x + y = 13$. Grade 5

Answer ..... **[3 marks]**

9.4 Solve $2x + 3y = 7$ and $3x + 5y = 11$. Grade 6

Answer ..... **[4 marks]**

9.5 Solve $5x + 2y = 8$ and $4x - 3y = 11$. Grade 6

Answer ..... **[4 marks]**

9.6 Three adult tickets and two child tickets cost £34. Two adult tickets and five child tickets cost £41. Work out the cost of an adult ticket and the cost of a child ticket. Grade 6

Answer ..... **[4 marks]**

9.7 Three pens and four pencils cost £2.90. Five pens and two pencils cost £3.20. Work out the cost of one pen and the cost of one pencil. Grade 6

Answer ..... **[4 marks]**

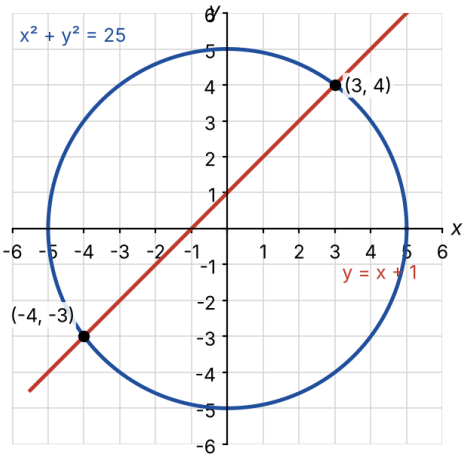
9.8 Solve $y = x + 2$ and $y = x^2 - 4$. Grade 7

Answer ..... **[4 marks]**

9.9 Solve $y = x^2 - 3x$ and $y = 2x - 6$. Grade 7

Answer ..... **[4 marks]**

9.10 The circle $x^2 + y^2 = 25$ and the line $y = x + 1$ are shown. Solve the simultaneous equations $x^2 + y^2 = 25$ and $y = x + 1$ algebraically. Grade 8



Answer ..... **[5 marks]**

9.11 Solve the simultaneous equations $x^2 + y^2 = 10$ and $y = 3x$. Grade 8

Answer ..... **[4 marks]**

9.12 Show that the line $y = x + 5$ is a tangent to the curve $y = x^2 - x + 6$. Grade 8

Answer ..... **[3 marks]**

9.13 The line $y = 2x + k$ is a tangent to the circle $x^2 + y^2 = 20$. Find the two possible values of k. Grade 9

Answer ..... **[5 marks]**

10 Algebraic proof

Grades 5 to 9 | 45 marks

Concepts explained

What is an algebraic proof?

A proof shows that a statement is true for **every** case, not just the ones you tried. Substituting numbers (1, 2, 3 …) is never a proof. Instead we use letters to represent **any** number of the stated type and show, with algebra, that the result must have the property claimed.

Building blocks: representing numbers

- **Any integer:** n. **Consecutive integers:** n, n + 1, n + 2, …
- **Even number:** 2n. **Odd number:** 2n + 1 (or 2n − 1). **Consecutive even numbers:** 2n, 2n + 2. **Consecutive odd numbers:** 2n + 1, 2n + 3.
- **Multiple of k:** k × (some integer), for example 3n is a multiple of 3. If you are using two different numbers, use **different letters** (a and b, or m and n).
- Two consecutive numbers squared: n^2 and $(n + 1)^2$.

The standard structure

Step 1: write the expression for what the question describes. **Step 2:** expand and simplify. **Step 3:** factorise so that the required factor appears (2 × something for even, 3 × something for a multiple of 3, 8 × something for a multiple of 8). **Step 4:** write a concluding sentence: "this is 2 × an integer, so it is even" or "this is a multiple of 8". The conclusion is worth a mark.

"Show that" and identities

To **show that** an expression equals a given result, start with one side, expand and simplify, and end with exactly what the question gives. Do not work from both sides at once and do not just state the answer. The symbol ≡ means "is identical to" and is true for all x.

Useful identities (these appear again and again)

- $(a + b)^2 - (a - b)^2 = 4ab$. * $a^2 - b^2 = (a - b)(a + b)$.
- The product of consecutive integers n(n + 1) is always **even**, because one of the two is even.
- The square of any integer is never 2 more than a multiple of 3, nor 2 more than a multiple of 4; the square of an odd number is 1 more than a multiple of 8.

Disproof by counter-example

To **disprove** a statement, you only need one example that breaks it. Find a value that makes the statement false and show the working. For "$n^2 + n + 41$ is always prime", n = 40 gives $40^2 + 40 + 41 = 1681 = 41^2$, which is not prime. One counter-example is a full disproof, but any number of examples that work is not a proof.

Formulae and rules

Formula	Statement
Even / odd	even: 2n; odd: 2n + 1; consecutive odds: 2n + 1, 2n + 3
Consecutive integers	n, n + 1, n + 2
Difference of two squares	$a^2 - b^2 = (a - b)(a + b)$
Always true	n(n + 1) is even; $(2n + 1)^2 = 4n(n + 1) + 1$
Perfect square	$(a - b)^2 \geq 0$ for all real a and b

Key facts

- Use letters for general cases and different letters for different numbers.
- Finish every proof with a sentence that states the conclusion in words.
- A proof must work for every value; examples can only disprove.
- Always factorise at the end to show the required factor.

Worked examples

Example 1. Prove that the sum of two consecutive integers is odd.

Let the integers be n and n + 1
n + (n + 1) = 2n + 1
2n is even, so 2n + 1 is odd. **Hence the sum of two consecutive integers is always odd.**

Example 2. Prove that the sum of three consecutive integers is a multiple of 3.

n + (n + 1) + (n + 2) = 3n + 3
= 3(n + 1)
This is 3 × an integer, so it is a multiple of 3.

Example 3. Show that $(n + 3)^2 - (n - 3)^2$ is a multiple of 12 for every integer n.

Expand: $(n^2 + 6n + 9) - (n^2 - 6n + 9) = 12n$
12n is 12 × an integer, so it is a multiple of 12.
Shortcut using the difference of squares: $[(n + 3) - (n - 3)][(n + 3) + (n - 3)] = 6 \times 2n = 12n$

Example 4. Prove that the product of two odd numbers is odd.

Let the numbers be 2a + 1 and 2b + 1 (different letters)
(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1
= 2(2ab + a + b) + 1
This is 2 × an integer plus 1, so it is odd.

Example 5. Prove that the square of any odd number is 1 more than a multiple of 8.

Let the odd number be 2n + 1. Then $(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$
n(n + 1) is the product of consecutive integers, so one of them is even and n(n + 1) is even
So 4n(n + 1) = 4 × an even number = a multiple of 8
Hence $(2n + 1)^2$ is 1 more than a multiple of 8.

Example 6. Prove that the sum of the squares of two consecutive integers is odd.

$n^2 + (n + 1)^2 = n^2 + n^2 + 2n + 1 = 2n^2 + 2n + 1$
$= 2(n^2 + n) + 1$
2 × an integer + 1 is odd.

Example 7. Show that $1/n - 1/(n + 1) = 1/[n(n + 1)]$.

Common denominator n(n + 1): $[(n + 1) - n] / [n(n + 1)]$
= **$1/[n(n + 1)]$**

Example 8. Jon says "$n^2 + n + 41$ is always a prime number". Show that Jon is wrong.

Try n = 40: $40^2 + 40 + 41 = 1600 + 81 = 1681$
1681 = 41 × 41, which is not prime
So the statement is false (n = 40 is a counter-example).

Common mistakes to avoid

- Using numbers (1, 2, 3 …) and calling it a proof.
- Using the same letter n for two different numbers, for example (2n + 1)(2n + 1) when the numbers are different.
- Stopping at the simplified expression and not stating why it has the required property.
- Writing even as n + 2 instead of 2n.
- Claiming 2n + 1 is odd without saying n is an integer.

Practice questions: 10. Algebraic proof

10.1 n is an integer. Write down an expression for Grade 5

(a) an even number **[1 mark]**

Answer .....

(b) the next odd number after 2n + 1 **[1 mark]**

Answer .....

(c) the sum of three consecutive integers n, n + 1 and n + 2, in its simplest form **[1 mark]**

Answer .....

10.2 Prove that the sum of two consecutive integers is always odd. Grade 6

Answer ..... **[3 marks]**

10.3 Prove that the sum of three consecutive integers is a multiple of 3. Grade 6

Answer ..... **[3 marks]**

10.4 Prove that the sum of any two consecutive odd numbers is a multiple of 4. Grade 7

Answer ..... **[3 marks]**

10.5 Grade 7

(a) Show that $(n + 3)^2 - (n - 3)^2 = 12n$. **[3 marks]**

Answer .....

(b) Explain why $(n + 3)^2 - (n - 3)^2$ is a multiple of 12 for every integer n. **[1 mark]**

Answer .....

10.6 Prove that the product of an odd number and an even number is always even. Grade 7

Answer ..... **[3 marks]**

10.7 Prove that the sum of the squares of two consecutive integers is always odd. Grade 7

Answer ..... **[3 marks]**

10.8 Grade 7

(a) Write down an example to show that the sum of two prime numbers is not always even. **[1 mark]**

Answer .....

(b) Jon says "$n^2 + n + 41$ is always a prime number". Show that Jon is wrong. **[2 marks]**

Answer .....

10.9 Prove that the difference between the squares of any two consecutive odd numbers is a multiple of 8. Grade 8

Answer ..... **[3 marks]**

10.10 Prove that the square of any odd number is 1 more than a multiple of 8. Grade 8

Answer ..... **[4 marks]**

10.11 Show that $1/n - 1/(n + 1) = 1/[n(n + 1)]$. Grade 8

Answer ..... **[3 marks]**

10.12 Show that $(2n + 3)^2 - (2n - 1)^2$ is always a multiple of 8. Grade 8

Answer ..... **[3 marks]**

10.13 Prove that the sum of the squares of any three consecutive integers is never a multiple of 3. Grade 9

Answer ..... **[4 marks]**

10.14 Prove that $x^2 + y^2 \geq 2xy$ for all real numbers x and y.

Answer ..... **[3 marks]**

11 Functions

Grades 4 to 9 | 46 marks

Concepts explained

What is a function?

A function is a rule that takes an **input** and gives exactly one **output**. We write f(x) = 3x − 2, read "f of x equals 3x − 2". It is a function machine: x goes in, "multiply by 3 then subtract 2" happens, f(x) comes out. The letter can be anything (g, h, ...) and the variable does not have to be x.

Evaluating and solving

f(4) means "replace x by 4": f(4) = 3 × 4 − 2 = 10. Use brackets for negatives: f(−1) = 3(−1) − 2 = −5. To **solve f(x) = 19** set the expression equal to 19 and solve: 3x − 2 = 19 gives x = 7. To solve f(x) = g(x) set the two expressions equal.

Composite functions

fg(x) means "do g first, then f": fg(x) = f(g(x)). Work from the **inside** out. With f(x) = 2x + 3 and g(x) = x^2: fg(x) = f(x^2) = $2x^2$ + 3, while gf(x) = g(2x + 3) = $(2x + 3)^2$ = $4x^2$ + 12x + 9. **fg and gf are usually different**: order matters. ff(x) means apply f twice.

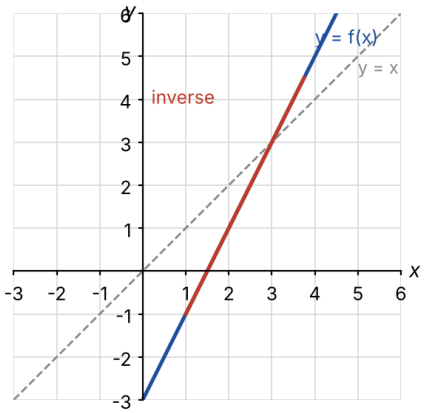
Inverse functions

The inverse function f^{-1} undoes f. If f(x) = 2x + 5, f^{-1} takes the output back to the input. **To find it:** (1) write y = f(x); (2) rearrange to make x the subject; (3) replace y by x. For y = 2x + 5: x = (y − 5)/2, so f^{-1}(x) = (x − 5)/2. Key facts: f(f^{-1}(x)) = x and f^{-1}(f(x)) = x. The graph of f^{-1} is the reflection of the graph of f in the line y = x. For a function with x in two places such as f(x) = (3x + 1)/(x − 2), use the "subject appears twice" method from section 4. A function that is its own inverse is called **self-inverse**: f(f(x)) = x.

Domain and range (Higher)

The **domain** is the set of allowed inputs. The **range** is the set of outputs. For f(x) = 1/(x − 3) the domain excludes x = 3 (division by zero). For g(x) = √(x − 2) the domain is x ≥ 2 (no square root of a negative). Always check the domain before accepting an answer to an equation.

f(x) = 2x − 3 and its inverse: reflection in y = x



Formulae and rules

Formula	Statement
Evaluate	f(a): replace x by a
Composite	fg(x) = f(g(x)): do g first

Inverse	$y = f(x) \rightarrow$ make x the subject $\rightarrow f^{-1}(x)$
Inverse property	$f(f^{-1}(x)) = x = f^{-1}(f(x))$
Self-inverse	$f(f(x)) = x$
Graph of inverse	reflection of $y = f(x)$ in the line $y = x$

Key facts

- fg(x): apply g first. Say "f after g". Always write the composition by substituting g(x) in place of x in f.
- $f^{-1}(x)$ does NOT mean 1/f(x). The −1 is not a power.
- Check an inverse: pick a number, apply f, then f^{-1}. You should return to the number you started with.
- If a denominator can be zero, state the excluded value.

Worked examples

Example 1. $f(x) = 3x - 2$. Work out f(4), f(−1) and solve $f(x) = 19$.

$f(4) = 3 \times 4 - 2 =$ **10**
$f(-1) = 3(-1) - 2 =$ **−5**
$3x - 2 = 19$ so $3x = 21$ and **$x = 7$**

Example 2. $f(x) = 2x + 3$ and $g(x) = x^2$. Find fg(x) and gf(x).

$fg(x) = f(x^2) = 2x^2 + 3$
$gf(x) = g(2x + 3) = (2x + 3)^2 =$ **$4x^2 + 12x + 9$**
$fg(x) = 2x^2 + 3$

Example 3. $f(x) = 2x + 5$. Find $f^{-1}(x)$.

$y = 2x + 5$, so $y - 5 = 2x$
$x = (y - 5)/2$
$f^{-1}(x) = (x - 5)/2$
Check: $f(3) = 11$ and $f^{-1}(11) = (11 - 5)/2 = 3$ (check)

Example 4. $f(x) = (3x + 1)/(x - 2)$. Which value of x cannot be used? Find $f^{-1}(x)$.

Denominator zero at **$x = 2$**
$y(x - 2) = 3x + 1$, so $xy - 2y = 3x + 1$, $xy - 3x = 1 + 2y$, $x(y - 3) = 1 + 2y$
$f^{-1}(x) = (2x + 1)/(x - 3)$

Example 5. $f(x) = 2x - 1$ and $g(x) = x^2$. Solve $fg(x) = gf(x)$.

$fg(x) = 2x^2 - 1$ and $gf(x) = (2x - 1)^2 = 4x^2 - 4x + 1$
$2x^2 - 1 = 4x^2 - 4x + 1$ so $2x^2 - 4x + 2 = 0$
$2(x - 1)^2 = 0$, **$x = 1$**

Example 6. State the domain of $g(x) = \sqrt{(x - 2)}$ and of $h(x) = 1/(x - 3)$.

$x - 2$ must be ≥ 0, so **domain of g: $x \geq 2$**
Domain of h: all x except $x = 3$

Example 7. $f(x) = (x + 3)/(x - 1)$. Show that f is its own inverse.

$f(f(x)) = [\ (x + 3)/(x - 1) + 3\ ] / [\ (x + 3)/(x - 1) - 1\ ]$
Multiply top and bottom by $(x - 1)$: $(x + 3 + 3x - 3) / (x + 3 - x + 1) = 4x/4$
$=$ **x**, so $f(f(x)) = x$ and $f = f^{-1}$

Example 8. f(x) = 2x − 3. Find $f^{-1}(5)$ and explain its relation to the graph of y = f(x).

$f^{-1}(x) = (x + 3)/2$, so $f^{-1}(5) =$ **4**

The graphs of f and f^{-1} are reflections of each other in y = x, so (4, 5) on f gives (5, 4) on f^{-1}

Common mistakes to avoid

- Doing fg in the wrong order. In fg(x), g is applied first.
- Treating $f^{-1}(x)$ as 1/f(x).
- Forgetting that $f(x) = 2x^2 + 3$ means substitute the whole of g(x) for x, using brackets.
- Forgetting brackets: f(−2) for $f(x) = x^2 + 3x$ is 4 − 6 = −2, not −4 − 6.
- Swapping x and y but not rearranging, or rearranging but forgetting to rename y as x in the answer.

Practice questions: 11. Functions

11.1 $f(x) = 4x - 7$. Grade 4

(a) Work out f(5). **[1 mark]**

Answer .....

(b) Work out f(−2). **[1 mark]**

Answer .....

(c) Solve f(x) = 21. **[2 marks]**

Answer .....

11.2 $g(x) = x^2 + 3x$. Grade 6

(a) Work out g(4) and g(−3). **[2 marks]**

Answer .....

(b) Solve g(x) = 10. **[3 marks]**

Answer .....

11.3 $f(x) = 2x + 3$ and $g(x) = x^2$. Find, in their simplest forms, Grade 6

(a) fg(x) **[2 marks]**

Answer .....

(b) gf(x) **[2 marks]**

Answer .....

11.4 $f(x) = 3x - 4$. Find $f^{-1}(x)$. Grade 6

Answer ..... **[2 marks]**

11.5 $f(x) = (x - 2)/5$. Find $f^{-1}(x)$. Grade 6

Answer ..... **[2 marks]**

11.6 $f(x) = x + 5$ and $g(x) = 3x$. Show that $fg(x) - gf(x) = -10$. Grade 7

Answer ..... **[3 marks]**

11.7 $f(x) = (3x + 1)/(x - 2)$. Grade 8

(a) Write down the value of x that cannot be used in this function. **[1 mark]**

Answer .....

(b) Find $f^{-1}(x)$. **[4 marks]**

Answer .....

11.8 $f(x) = 2x - 1$ and $g(x) = x^2$. Solve $fg(x) = gf(x)$. Grade 8

Answer ..... **[4 marks]**

11.9 $f(x) = x^2 - 4x$ and $g(x) = x + 1$. Solve $fg(x) = 0$. Grade 8

Answer ..... **[4 marks]**

11.10 $f(x) = \sqrt{(x - 3)}$. Grade 8

(a) Write down the domain of f. **[1 mark]**

Answer .....

(b) Solve $f(x) = 5$. **[2 marks]**

Answer .....

(c) Find $f^{-1}(x)$. **[2 marks]**

Answer .....

11.11 $f(x) = ax + b$. Given that $f(2) = 7$ and $f(-1) = -2$, find a and b and write down $f^{-1}(x)$. Grade 8

Answer ..... **[4 marks]**

11.12 $f(x) = (x + 3)/(x - 1)$ for $x \neq 1$. Show that f is its own inverse, that is $ff(x) = x$. Grade 9

Answer ..... **[4 marks]**

12 Iteration

Grades 5 to 9 | 43 marks

Concepts explained

The idea of iteration

Some equations cannot be solved exactly with the methods at GCSE (for example $x^3 - 3x - 5 = 0$). **Iteration** finds an **approximate** solution by repeating a calculation. An iteration formula gives the next approximation x_{n+1} from the current one x_n. You start from a starting value x_0 (or x_1) and repeat. If the values settle to a number, the iteration **converges** and that number is a root of the equation.

Using your calculator

Type the starting value and press =. Then type the formula using the **ANS** key in place of x_n, and keep pressing =. Write down at least 4 decimal places of every value and do not round during the calculation. For $x_{n+1} = \sqrt[3]{(3x_n + 5)}$ with $x_0 = 1$, enter 1 =, then $(3 \text{ ANS} + 5)^{1/3}$ =.

Showing a root lies in an interval (change of sign)

Rearrange the equation to f(x) = 0. Work out f(a) and f(b). If f(a) and f(b) have **opposite signs** (one positive, one negative) and the graph is continuous, then there is a root between a and b. For $f(x) = x^3 - 3x - 5$: f(2) = −3 and f(3) = 13, so there is a root between 2 and 3. State the values and the sign change in your answer.

Decimal search

To narrow a root to 1 d.p.: evaluate f at the tenths between a and b (2.1, 2.2, 2.3 …) and find where the sign changes. To confirm the rounding, test the halfway value (2.25). f(2.25) < 0 and f(2.3) > 0 so the root is between 2.25 and 2.3, which rounds to 2.3.

Rearranging an equation into an iterative form

The same equation can give several iteration formulae. To get one, rearrange so that a single x is the subject and the rest is a function of x. From $x^3 - 3x - 5 = 0$: $x^3 = 3x + 5$, so $x = \sqrt[3]{(3x + 5)}$, giving $x_{n+1} = \sqrt[3]{(3x_n + 5)}$. From $x^2 - 3x - 1 = 0$: divide by x after writing $x^2 = 3x + 1$, giving $x = 3 + 1/x$. Not every rearrangement converges.

Giving the answer to a required accuracy

Continue until two consecutive values agree to the accuracy you need. To state a root to 2 d.p., you need the iterates to settle at the same value to 2 d.p. (the third decimal place can still change). Then round the last value, and say: "the root is 2.28 to 2 d.p.".

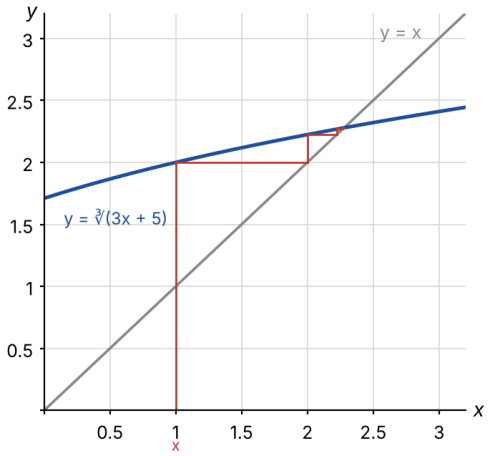
Cobweb and staircase diagrams (Higher)

Draw y = x and y = g(x). From x_0 on the x-axis go **vertically** to the curve, then **horizontally** to the line y = x, then vertically to the curve again. When the curve is flatter than the line (gradient between −1 and 1 near the root) the staircase (or spiral) moves in towards the intersection: convergence. If the curve is steeper than y = x, the steps move away: divergence.

Limit of an iteration

If $x_{n+1} = g(x_n)$ converges to L then L = g(L). Writing this out gives an equation for L. Reject limits that do not make sense, for example a negative value when each term is a square root.

Staircase for x_{n+1} = cube root of $(3x_n + 5)$, starting at $x_0 = 1$



Formulae and rules

Formula	Statement
Iteration	$x_{n+1} = g(x_n)$, starting at x_0
Change of sign	f(a) and f(b) opposite signs → a root between a and b
Limit	if $x_n \to L$ then $L = g(L)$
Staircase	vertical to the curve, horizontal to y = x, repeat

Key facts

- Keep all the digits on the calculator display until the very end; rounding in the middle can give a wrong third decimal place.
- For "show that" change-of-sign questions you must state both values and a conclusion.
- The starting value only determines which root you find (and if the iteration converges at all).
- A change of sign works only when f is continuous (for example it fails across the asymptote of 1/x).

Worked examples

Example 1. Show that the equation $x^3 - 3x - 5 = 0$ has a solution between x = 2 and x = 3.

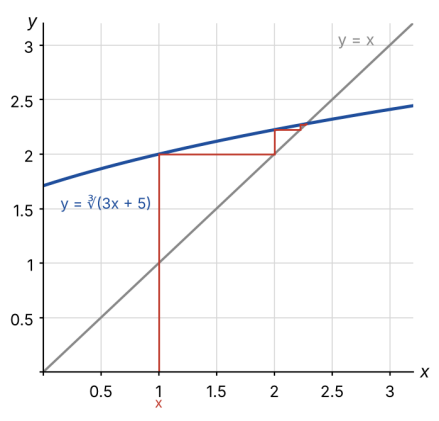
$f(x) = x^3 - 3x - 5$
$f(2) = 8 - 6 - 5 = -3$ (negative)
$f(3) = 27 - 9 - 5 = 13$ (positive)
There is a change of sign, so there is a root between 2 and 3.

Example 2. Show that $x^3 - 3x - 5 = 0$ can be rearranged to give $x = \sqrt[3]{(3x + 5)}$.

$x^3 = 3x + 5$
Take the cube root of both sides: **$x = \sqrt[3]{(3x + 5)}$**

Example 3. Use $x_{n+1} = \sqrt[3]{(3x_n + 5)}$ with $x_0 = 1$ to find a solution of $x^3 - 3x - 5 = 0$ to 2 decimal places.

$x_1 = \sqrt[3]{(3 \times 1 + 5)} = \sqrt[3]{8} = 2$
$x_2 = \sqrt[3]{11} = 2.2240$
$x_3 = 2.2684$, $x_4 = 2.2770$, $x_5 = 2.2786$, $x_6 = 2.2789$, $x_7 = 2.2790$
The values are settling at 2.279…, so the solution is **x = 2.28** (2 d.p.)



Example 4. Find the root of $x^3 - 3x - 5 = 0$ to 1 decimal place using a decimal search.

f(2.2) = 10.648 − 6.6 − 5 = −0.952 (negative)
f(2.3) = 12.167 − 6.9 − 5 = 0.267 (positive)
So the root is between 2.2 and 2.3. Test the middle: f(2.25) = 11.390625 − 6.75 − 5 = −0.359 (negative)
Root between 2.25 and 2.3, so **x = 2.3** (1 d.p.)

Example 5. $x_{n+1} = \sqrt{(2x_n + 3)}$ with $x_0 = 2$. Work out x_1, x_2 and x_3. What number do the terms approach?

$x_1 = \sqrt{7} = 2.6458$
$x_2 = \sqrt{(2 \times 2.6458 + 3)} = \sqrt{8.2915} = 2.8795$
$x_3 = 2.9596$
The values approach **3** (check: $3 = \sqrt{(2 \times 3 + 3)} = \sqrt{9}$ (check))

Example 6. $x_{n+1} = \sqrt{(x_n + 6)}$. The sequence converges to L. Find L exactly.

$L = \sqrt{(L + 6)}$
$L^2 = L + 6$, so $L^2 - L - 6 = 0$
$(L - 3)(L + 2) = 0$, so L = 3 or L = −2
Every term is a positive square root, so **L = 3**

Example 7. Show that $x^2 - 3x - 1 = 0$ can be written as $x = 3 + 1/x$. Use $x_{n+1} = 3 + 1/x_n$ with $x_0 = 3$ to find the positive solution to 2 d.p.

$x^2 = 3x + 1$; divide by x: **$x = 3 + 1/x$**
$x_1 = 3.3333$, $x_2 = 3.3$, $x_3 = 3.3030$, $x_4 = 3.3028$, $x_5 = 3.3028$
Positive solution **x = 3.30** (2 d.p.)

Common mistakes to avoid

- Rounding to 1 or 2 decimal places at each step. Keep full calculator accuracy.
- Stopping after only two iterations without checking the values have settled.
- In a change-of-sign argument, not stating the values or the conclusion.
- Giving the answer to the wrong accuracy: iterate until the digits you need stop changing.
- Entering the formula with the wrong bracket: $\sqrt[3]{(3 \text{ ANS} + 5)}$ needs brackets around the whole expression.

Practice questions: 12. Iteration

12.1 $x_{n+1} = 2x_n - 3$ and $x_1 = 5$. Work out x_2, x_3 and x_4. Grade 5

Answer ..... **[3 marks]**

12.2 Show that the equation $x^3 + x - 5 = 0$ has a solution between $x = 1$ and $x = 2$. Grade 6

Answer ..... **[2 marks]**

12.3 $x^3 + x - 5 = 0$ has a solution between 1 and 2. Grade 6

(a) Show that the equation can be rearranged to give $x = \sqrt[3]{(5 - x)}$. **[2 marks]**

Answer .....

(b) Use $x_{n+1} = \sqrt[3]{(5 - x_n)}$ with $x_0 = 1.5$ to find x_1, x_2 and x_3. Give each answer to 4 decimal places. **[3 marks]**

Answer .....

(c) Write down the solution of the equation to 2 decimal places. **[1 mark]**

Answer .....

12.4 $x_{n+1} = \sqrt{(2x_n + 3)}$ and $x_0 = 2$. Grade 6

(a) Work out x_1, x_2 and x_3 to 4 decimal places. **[3 marks]**

Answer .....

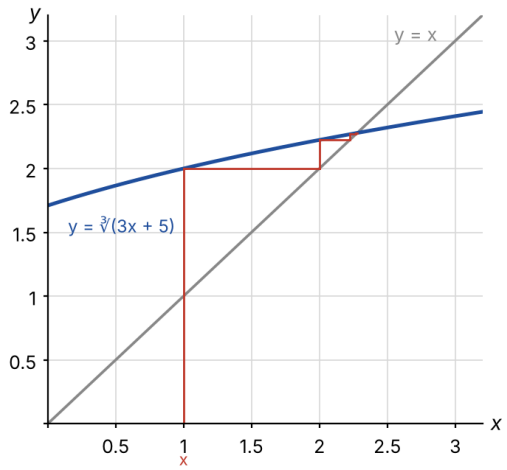
(b) The sequence approaches the solution of $x^2 = 2x + 3$. What is this solution (the one it approaches)? **[2 marks]**

Answer .....

12.5 Show that $x^2 - 3x - 1 = 0$ can be rearranged to give $x = 3 + 1/x$. Use $x_{n+1} = 3 + 1/x_n$ with $x_0 = 3$ to find the positive solution to 2 decimal places. Grade 7

Answer ..... **[5 marks]**

12.6 The diagram shows the graphs of $y = x$ and $y = \sqrt[3]{(3x + 5)}$. The iteration $x_{n+1} = \sqrt[3]{(3x_n + 5)}$ starts at $x_0 = 1$. Grade 7



(a) Write down the value of x_1. **[1 mark]**

Answer .....

(b) Does the iteration converge or diverge? Use the diagram to explain. **[2 marks]**

Answer .....

(c) Estimate the solution of $x^3 - 3x - 5 = 0$ from the diagram. **[1 mark]**

Answer .....

12.7 $x^3 - 3x - 5 = 0$ has a root between 2 and 3. Use a decimal search to find the root to 1 decimal place. Show your values of f(x). Grade 8

Answer ..... **[4 marks]**

12.8 Show that $x^3 + 4x - 1 = 0$ has a root between 0 and 1. Rearrange the equation to give $x = 1/(x^2 + 4)$. Use $x_{n+1} = 1/(x_n^2 + 4)$ with $x_0 = 0.2$ to find the root to 3 decimal places. Grade 8

Answer ..... **[5 marks]**

12.9 $x_{n+1} = \sqrt{(x_n + 6)}$. The sequence $x_0, x_1, x_2, \ldots$ converges to a limit L. Find the exact value of L. Grade 9

Answer ..... **[4 marks]**

12.10 Show that the iteration $x_{n+1} = 1/(x_n + 1)$ has a limit L satisfying $L^2 + L - 1 = 0$, and find L to 3 decimal places. Grade 9

Answer ..... **[5 marks]**

Notes

Use this page for corrections, questions to ask your tutor and formulae you keep forgetting.