Edexcel
Circle Theorem, Algebra and Surds Questions

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Every question has three pages: the question with working space, the marking page with a grade 9 answer, and the concept page.

Mark codes

These copy the style of Edexcel mark schemes. They are not an official scheme.

M1	Method: a correct step towards the answer, even with an arithmetic slip.
A1	Accuracy: a correct value or conclusion. Usually needs the method mark first.
C1	Communication: a correct reason, in words, next to the line it justifies.
B1	Independent: a correct fact or value that needs no method.

Proof rule: in a 'prove' or 'show that' question every line needs a reason, and the last line must state exactly what you were asked to prove.

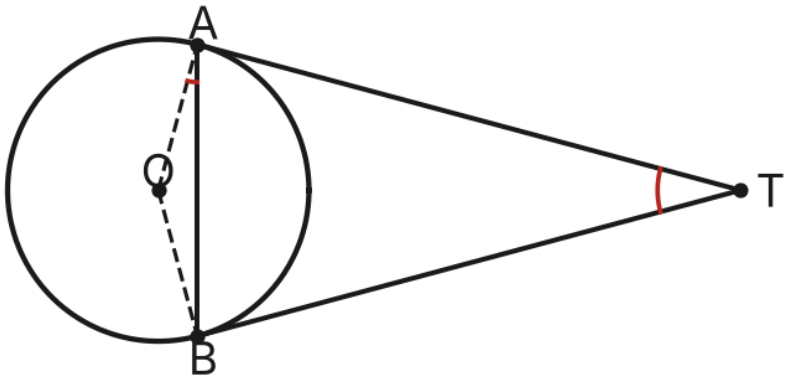
Question 1

4 marks

Circles: Algebra and angle chase

TA and TB are tangents to a circle, centre O. Angle ATB = (4x - 10)° and angle OAB = (x + 5)°.

Work out the value of x. Give reasons for your working. (4)

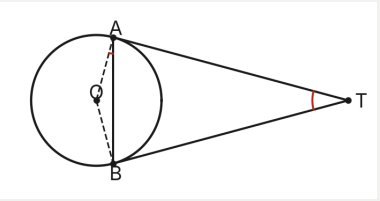


Working space

Question 1: marking and grade 9 answer

4 marks

TA and TB are tangents to a circle, centre O. Angle ATB = (4*x* - 10)° and angle OAB = (*x* + 5)°.
Work out the value of *x*. Give reasons for your working. (4)



Line of the answer	Mark	How it is marked
Angles OAT and OBT are 90° (tangent meets radius). In quadrilateral OATB, angle AOB = 360 - 90 - 90 - (4*x* - 10) = 190 - 4*x*	**M1**	Reason stated. Gets the centre angle in terms of x.
OA = OB (radii), so triangle OAB is isosceles: angle OAB = (180 - (190 - 4*x*)) ÷ 2 = 2*x* - 5	**M1**	Second reason, isosceles triangle. Now OAB is in terms of x.
Set the two expressions for OAB equal: *x* + 5 = 2*x* - 5	**M1**	The equation comes from two routes to the same angle.
x = 10. Check: ATB = 30°, AOB = 150°, OAB = 15°	**A1**	Correct value. The check is the habit.

The trap
Setting x + 5 = 4x - 10, as if the two angles were equal. They are not related that way. OAB is half of ATB.

The habit that beats it
Write the unknown angle two ways using the geometry, then equate. Substitute back to check.

A tangent meets the radius at 90°

In plain words

A tangent touches the circle at exactly one point. The radius drawn to that point of contact is always at a right angle to the tangent.

The theorem

Theorem: the angle between a tangent and the radius at the point of contact is 90°.
Two tangents from the same point T are equal in length, so TA = TB.

How to use it

1. Draw the radii OA and OB to the points of contact and mark both right angles.

2. Look for the kite OATB. Its angles add to 360°, so AOB = 180° - ATB.

3. Triangle OAB is isosceles because OA = OB (radii), so OAB = (180° - AOB) ÷ 2.

4. When two routes give the same angle, write it two ways and form an equation.

Worked mini-example
Tangent TA, centre O, angle ATO = 30°. Angle OAT = 90° (tangent and radius), so angle AOT = 180 - 90 - 30 = 60°.

Spot it

Any question with a tangent and the word 'centre'. The first line is always the right angle.

Question 2

4 marks

Algebra: completing the square

(a) Write $2x^2 - 12x + 7$ in the form $a(x - p)^2 + q$. (2)

(b) Hence solve $2x^2 - 12x + 7 = 0$. Give your solutions in surd form. (2)

Working space

Question 2: marking and grade 9 answer

4 marks

(a) Write $2x^2 - 12x + 7$ in the form $a(x - p)^2 + q$. (2)
(b) Hence solve $2x^2 - 12x + 7 = 0$. Give your solutions in surd form. (2)

Line of the answer	Mark	How it is marked
$2x^2 - 12x + 7 = 2(x^2 - 6x) + 7 = 2[(x - 3)^2 - 9] + 7$	**M1**	Factor 2 out of the x terms only, then complete the square inside the bracket.
$= 2(x - 3)^2 - 11$, so $a = 2$, $p = 3$, $q = -11$	**A1**	The -9 must be multiplied by 2 before 7 is added.
$2(x - 3)^2 = 11$, so $(x - 3)^2 = 11/2$	**M1**	Use the completed square form. No formula is needed.
$x - 3 = \pm\sqrt{(11/2)}$, so $x = 3 \pm \sqrt{22} / 2$	**A1**	Both roots, in simplest surd form: $\sqrt{(11/2)} = \sqrt{22} / 2$.

The trap
Writing $2(x - 6)^2 - 36 + 7$, which forgets that the 2 is outside the bracket and halves 12 instead of 6.

The habit that beats it
Take the coefficient of x^2 out of the x terms first. Then halve the new coefficient of x. Expand your answer back to check.

Completing the square

In plain words

Completing the square rewrites a quadratic so that x appears only once. That lets you solve it with a square root, find its turning point, or both.

The formula or method

$x^2 + bx + c = (x + b/2)^2 + c - (b/2)^2$
If the coefficient of x^2 is a: $a x^2 + bx + c = a(x + b/2a)^2 + c - b^2/4a$
Turning point of $a(x - p)^2 + q$ is (p, q).

How to use it

1. Factor out a from the x^2 and x terms only. Leave the constant outside.

2. Halve the new x coefficient, put it in the bracket, subtract its square inside the bracket.

3. Multiply the subtracted square by a, then combine with the constant.

4. To solve: move the constant across, divide by a, take ± the square root.

Worked mini-example
$x^2 + 8x + 3 = (x + 4)^2 - 16 + 3 = (x + 4)^2 - 13$. The minimum point is (-4, -13).

Spot it

'Write in the form $a(x + p)^2 + q$', 'find the turning point', or 'solve, giving your answer in surd form'.

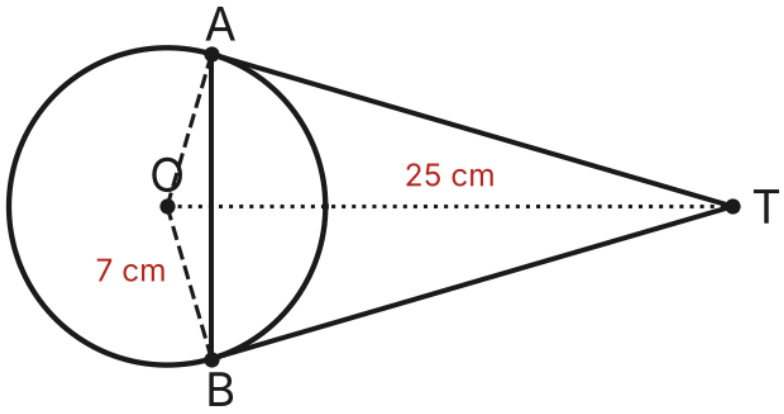
Question 3

4 marks

Circles: Calculation

T is 25 cm from the centre O of a circle of radius 7 cm. TA and TB are tangents to the circle.

Work out the length of the chord AB. (4)

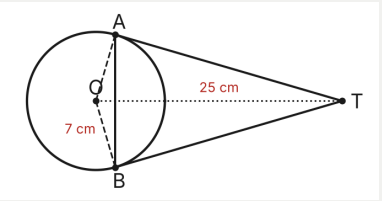


Working space

Question 3: marking and grade 9 answer

4 marks

T is 25 cm from the centre O of a circle of radius 7 cm. TA and TB are tangents to the circle.
Work out the length of the chord AB. (4)



Line of the answer	Mark	How it is marked
Angle OAT = 90° (tangent meets radius). $AT^2 = 25^2 - 7^2 = 576$, so AT = 24 cm	**M1**	Pythagoras in triangle OAT with hypotenuse 25.
OATB is a kite, so OT meets AB at its midpoint M at right angles (symmetry)	**C1**	The reason that lets you work with half of AB.
Area of triangle OAT: ½ × 7 × 24 = ½ × 25 × AM, so AM = 168 ÷ 25 = 6.72	**M1**	The same area two ways. Fastest route to the height.
AB = 2 × 6.72 = **13.44 cm**	**A1**	Double the half-chord.

The trap
Giving AB = 24 (the tangent length) or 14 (the diameter). The chord AB is shorter than both.

The habit that beats it
Area of the same triangle two ways gives a perpendicular height. Alternative: triangles OAM and OTA are similar.

Tangents from a point and the kite

In plain words

From a point outside a circle you can draw two tangents. They are the same length, and together with the two radii they form a kite that is split by a line of symmetry.

The theorem

Tangent length: $TA^2 = OT^2 - r^2$ (right angle at A).
The line OT is a line of symmetry: it bisects AB and meets it at 90°.
Area of triangle OAT = ½ × OA × AT = ½ × OT × AM, where M is the midpoint of AB.

How to use it

1. Mark the right angle at A and use Pythagoras for the tangent length.

2. Name the midpoint M of AB. Say that OATB is a kite, so AM is perpendicular to OT.

3. Write the area of triangle OAT two ways and solve for AM.

4. Double AM to get the whole chord.

Worked mini-example
T is 10 cm from O and the radius is 6 cm. TA = √(100 - 36) = 8 cm. AM = (6 × 8) / 10 = 4.8, so AB = 9.6 cm.

Spot it

The words 'tangents from T' together with a distance OT. Think kite, right angle, area.

Question 4

4 marks

Surds: rationalising the denominator

Show that $(3 + \sqrt{5}) / (2 - \sqrt{5})$ can be written in the form $a + b\sqrt{5}$, where a and b are integers.

Find the values of a and b. (4)

Working space

Question 4: marking and grade 9 answer

4 marks

Show that $(3 + \sqrt{5}) / (2 - \sqrt{5})$ can be written in the form $a + b\sqrt{5}$, where a and b are integers.
Find the values of a and b. (4)

Line of the answer	Mark	How it is marked
Multiply top and bottom by the conjugate: $(3 + \sqrt{5})(2 + \sqrt{5}) / (2 - \sqrt{5})(2 + \sqrt{5})$	**M1**	The conjugate of $2 - \sqrt{5}$ is $2 + \sqrt{5}$.
Numerator: $6 + 3\sqrt{5} + 2\sqrt{5} + 5 = 11 + 5\sqrt{5}$	**M1**	Four terms expanded, with $\sqrt{5} \times \sqrt{5} = 5$.
Denominator: $2^2 - (\sqrt{5})^2 = 4 - 5 = -1$	**B1**	Difference of two squares leaves no surd.
$(11 + 5\sqrt{5}) / (-1) = -11 - 5\sqrt{5}$, so $a = -11$ and $b = -5$	**A1**	Divide every term by -1, including the surd term.

The trap
Dividing only 11 by -1 and leaving $+5\sqrt{5}$, or multiplying the top by $2 + \sqrt{5}$ and the bottom by $2 - \sqrt{5}$.

The habit that beats it
Whatever you multiply the bottom by, you multiply the top by. Write both brackets out in full.

Rationalising the denominator

In plain words

A fraction with a surd on the bottom is hard to work with. Multiplying top and bottom by the conjugate removes the surd from the bottom without changing the value.

The formula or method

$(a + b)(a - b) = a^2 - b^2$
The conjugate of $p + q\sqrt{r}$ is $p - q\sqrt{r}$.
$(\sqrt{a})(\sqrt{a}) = a$. In general $\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$.

How to use it

1. Write the conjugate of the denominator.

2. Multiply the numerator by it, expanding all four terms.

3. Multiply the denominator by it. The middle terms cancel, leaving a whole number.

4. Divide every term on top by that number and simplify.

Worked mini-example
$1 / (1 + \sqrt{2}) = (1 - \sqrt{2}) / (1 - 2) = (1 - \sqrt{2}) / (-1) = \sqrt{2} - 1$.

Spot it

'Rationalise', 'show that it can be written in the form $a + b\sqrt{c}$', or an answer with a surd below the line.

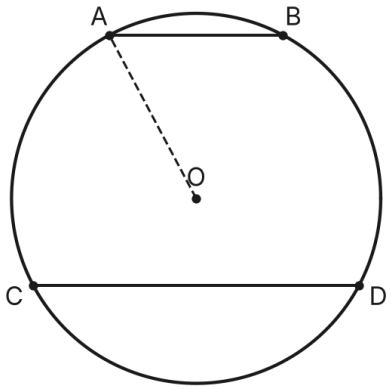
Question 5

4 marks

Circles: Calculation, two cases

A circle has radius 17 cm. AB and CD are parallel chords. AB = 16 cm and CD = 30 cm.

Work out the two possible distances between AB and CD. (4)

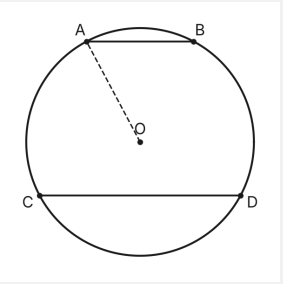


Working space

Question 5: marking and grade 9 answer

4 marks

A circle has radius 17 cm. AB and CD are parallel chords. AB = 16 cm and CD = 30 cm.
Work out the two possible distances between AB and CD. (4)



Line of the answer	Mark	How it is marked
The perpendicular from the centre bisects a chord, so the half-chords are 8 cm and 15 cm	**C1**	Chord theorem stated. Gives the legs of the right-angled triangles.
Distance from O to AB = $\sqrt{(17^2 - 8^2)} = 15$. Distance from O to CD = $\sqrt{(17^2 - 15^2)} = 8$	**M1**	Pythagoras twice. Hypotenuse is the radius 17.
Chords on opposite sides of O: 15 + 8 = **23 cm**	**A1**	First case.
Chords on the same side of O: 15 - 8 = **7 cm**	**A1**	Second case. This is a full mark that most students never reach.

The trap
Finding 23 and stopping. The question says 'two possible distances', and the second configuration is a full mark.

The habit that beats it
Whenever a question says 'possible' or 'all', draw both configurations before you calculate.

A perpendicular from the centre bisects a chord

In plain words

Draw a line from the centre to a chord so that it meets the chord at a right angle. It cuts the chord in half. This gives right-angled triangles with the radius as hypotenuse.

The theorem

Theorem: the perpendicular from the centre of a circle to a chord bisects the chord.
Distance from the centre to the chord: $d = \sqrt{(r^2 - (\text{half the chord})^2)}$.

How to use it

1. Draw the radius to one end of the chord and the perpendicular from O.

2. Half-chord, distance and radius form a right-angled triangle. Use Pythagoras.

3. If the chords are parallel, the centre can lie between them or outside them.

4. For two cases, find the distance to each chord and both the sum and the difference.

Worked mini-example
Radius 5 cm, chord 8 cm. Half-chord = 4, distance from O = $\sqrt{(25 - 16)} = 3$ cm.

Spot it

The words 'chord', 'parallel' and 'possible'. Ask whether both chords can be on the same side of the centre.

Question 6

4 marks

Algebra: the discriminant

The equation $kx^2 - 4x + 1 = 0$, where k is a constant, has two distinct real roots.

Find the range of possible values of k. (4)

Working space

Question 6: marking and grade 9 answer

4 marks

The equation $kx^2 - 4x + 1 = 0$, where k is a constant, has two distinct real roots.
Find the range of possible values of k. (4)

Line of the answer	Mark	How it is marked
$b^2 - 4ac = (-4)^2 - 4(k)(1) = 16 - 4k$	**M1**	Correct substitution with a = k, b = -4, c = 1.
Two distinct real roots, so $b^2 - 4ac > 0$: $16 - 4k > 0$	**M1**	Strictly greater than zero. Equal to zero would give one repeated root.
$k < 4$	**A1**	Rearrange: 16 > 4k.
If $k = 0$ the equation is 'linear' with only one root, so $k \neq 0$. Final answer: $k < 4$, $k \neq 0$	**C1**	The reason k cannot be 0 is what earns this mark.

The trap
Stopping at k < 4. That includes k = 0, where the equation is not a quadratic and has only one root.

The habit that beats it
Ask what makes the equation a quadratic. The coefficient of x^2 must not be zero.

The discriminant

In plain words

The discriminant tells you how many real solutions a quadratic has without solving it. It is the part under the root in the quadratic formula.

The formula or method

For $ax^2 + bx + c = 0$: discriminant = $b^2 - 4ac$.
$b^2 - 4ac > 0$: two distinct real roots. $b^2 - 4ac = 0$: one repeated root. $b^2 - 4ac < 0$: no real roots.
A line is a tangent to a curve when the discriminant of the combined equation is 0.

How to use it

1. Identify a, b and c exactly, including signs and any unknown such as k.

2. Write $b^2 - 4ac$, then simplify.

3. Write the correct inequality or equals sign for the case the question describes.

4. Solve the inequality. Check that a is not 0, since otherwise it is not a quadratic.

Worked mini-example
$x^2 + 6x + k = 0$ has two distinct roots: $36 - 4k > 0$, so $k < 9$.

Spot it

'Distinct', 'equal roots', 'no real roots', 'touches' or 'tangent'. Each translates to a statement about $b^2 - 4ac$.

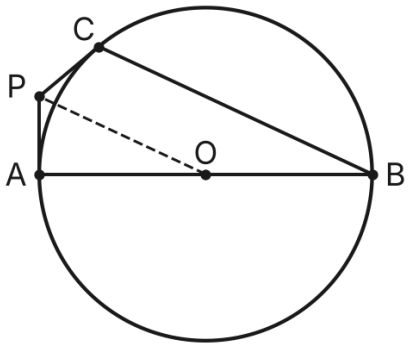
Question 7

4 marks

Circles: Proof

AB is a diameter of a circle, centre O. The tangents at A and at C meet at P.

Prove that OP is parallel to BC. (4)

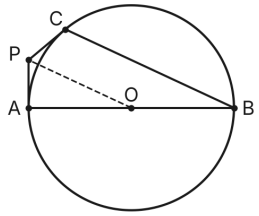


Working space

Question 7: marking and grade 9 answer

4 marks

AB is a diameter of a circle, centre O. The tangents at A and at C meet at P.
Prove that OP is parallel to BC. (4)



Line of the answer	Mark	How it is marked
PA = PC (tangents from an external point) and OA = OC (radii), so OAPC is a kite and OP is perpendicular to AC	**C1**	Two equal pairs with reasons, then the kite diagonal property.
Angle ACB = 90° (angle in a semicircle), so BC is perpendicular to AC	**C1**	The second reason. AB is a diameter.
OP and BC are both perpendicular to AC	**M1**	Pulling the two facts together.
So OP is parallel to BC (lines perpendicular to the same line are parallel)	**A1**	Conclusion with the rule named.

The trap
Saying OP is perpendicular to AC with no reason. 'It looks perpendicular' scores zero.

The habit that beats it
A kite is two congruent triangles. Say 'kite' and say why it is one.

Angle in a semicircle and perpendicular lines

In plain words

A triangle drawn on a diameter always has a right angle at the third point. Two lines that are each perpendicular to the same line must be parallel to each other.

The theorem

Theorem: the angle in a semicircle is 90°.
Two lines perpendicular to the same line are parallel.
Tangents from a point are equal, and OA = OC (radii), so OP is the perpendicular bisector of AC.

How to use it

1. Mark the 90° angle on the diameter, with its reason.

2. Show the other line is perpendicular to the same line. Use equal lengths on both sides (a kite).

3. State the conclusion as a rule: both are perpendicular to AC, so they are parallel.

4. In a proof, every line needs a reason beside it.

Worked mini-example
AB is a diameter and angle ABC = 35°. Angle ACB = 90° (angle in a semicircle), so angle BAC = 180 - 90 - 35 = 55°.

Spot it

The word 'diameter' in a question that mentions a third point on the circle.

Question 8

4 marks

Surds: equation with a root

Solve $\sqrt{(2x + 1)} = x - 1$.

Check each solution and say whether it is valid. (4)

Working space

Question 8: marking and grade 9 answer

4 marks

Solve $\sqrt{(2x + 1)} = x - 1$.
Check each solution and say whether it is valid. (4)

Line of the answer	Mark	How it is marked
Square both sides: $2x + 1 = (x - 1)^2 = x^2 - 2x + 1$	**M1**	The right side must be expanded as a full bracket, not as $x^2 - 1$.
$x^2 - 4x = 0$, so $x(x - 4) = 0$	**M1**	Collect to zero and factorise.
$x = 0$ or $x = 4$	**A1**	Both roots of the squared equation.
Check $x = 0$: left side = 1, right side = -1, so it is not valid. Check $x = 4$: $\sqrt{9} = 3 = 4 - 1$, valid. Answer: $x = 4$	**C1**	Squaring creates extra solutions. The check is the mark.

The trap
Giving x = 0 and x = 4 as the answer. A square root cannot equal -1.

The habit that beats it
After squaring, always substitute each answer back into the original equation.

Solving equations with a square root

In plain words

To remove a square root you square both sides. That can create answers that do not satisfy the original equation, so you always check.

The formula or method

Isolate the root on one side before squaring.
$(x - a)^2 = x^2 - 2ax + a^2$, a full three-term expansion.
A square root symbol means the positive root only.

How to use it

1. Get the root on its own on one side.

2. Square both sides and expand fully.

3. Rearrange to a quadratic equal to zero and solve.

4. Substitute each answer into the original equation and reject any that fail.

Worked mini-example
$\sqrt{(x + 2)} = x$ gives $x + 2 = x^2$, so $x^2 - x - 2 = 0$ and $x = 2$ or $x = -1$. Check: $x = 2$ gives $\sqrt{4} = 2$, valid. $x = -1$ gives $\sqrt{1} = -1$, false, so reject it.

Spot it

A root on one side and an expression in x on the other. The question may say 'check your solutions'.

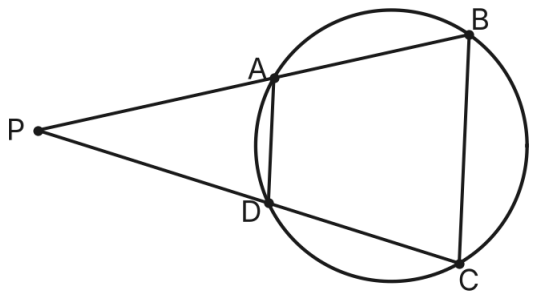
Question 9

4 marks

Circles: Proof with similar triangles

ABCD is a cyclic quadrilateral. BA and CD are extended to meet at P, with A between P and B and D between P and C.

Prove that triangles PAD and PCB are similar, and hence that PA × PB = PC × PD. (4)

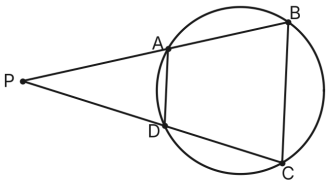


Working space

Question 9: marking and grade 9 answer

4 marks

ABCD is a cyclic quadrilateral. BA and CD are extended to meet at P, with A between P and B and D between P and C.
Prove that triangles PAD and PCB are similar, and hence that PA × PB = PC × PD. (4)



Line of the answer	Mark	How it is marked
Angle APD = angle CPB (common angle)	**M1**	Both triangles share the angle at P.
Angle PAD = 180° - angle DAB (angles on a straight line) and angle DAB + angle DCB = 180° (opposite angles in a cyclic quadrilateral), so angle PAD = angle DCB = angle PCB	**C1**	Two reasons: straight line, then cyclic quadrilateral.
Two pairs of equal angles, so triangles PAD and PCB are similar	**C1**	Similarity must be justified by the equal angles.
PA ÷ PC = PD ÷ PB, so PA × PB = PC × PD	**A1**	Corresponding sides: P to P, A to C, D to B.

The trap
Matching the vertices in the wrong order (P to P, A to B). The ratios then come out wrong.

The habit that beats it
Write the similarity with the vertices in matching order before writing any ratio.

Cyclic quadrilaterals and similar triangles

In plain words

A cyclic quadrilateral has all four corners on a circle. Its angles obey two rules. Extending its sides can create similar triangles, which give equal ratios of lengths.

The theorem

Opposite angles of a cyclic quadrilateral add to 180°.
An exterior angle equals the interior opposite angle.
Intersecting secants: PA × PB = PC × PD.

How to use it

1. Mark the angle shared by both triangles. It is the same angle, so it counts as one pair.

2. Use the exterior angle rule to match a second pair of angles. Then say that the triangles are similar (AA).

3. Write the ratio of corresponding sides, with matching letters in matching positions.

4. Cross-multiply to get PA × PB = PC × PD.

Worked mini-example
PA = 3, PB = 8 and PD = 4. PA × PB = 24 = PC × PD, so PC = 6.

Spot it

A cyclic quadrilateral with two sides extended to meet at a point outside the circle.

Question 10

6 marks

Algebra: equation with fractions

Solve $3 / (x + 1) + 2 / (x - 2) = 1$

Give your solutions in the form $a \pm \sqrt{b}$. (6)

Working space

Question 10: marking and grade 9 answer

6 marks

Solve $3 / (x + 1) + 2 / (x - 2) = 1$
Give your solutions in the form $a \pm \sqrt{b}$. (6)

Line of the answer	Mark	How it is marked
Multiply through by $(x + 1)(x - 2)$: $3(x - 2) + 2(x + 1) = (x + 1)(x - 2)$	**M1**	Every term is multiplied, including the 1 on the right side.
Left side: $3x - 6 + 2x + 2 = 5x - 4$	**A1**	Expand and collect.
Right side: $x^2 - x - 2$	**M1**	Expand the product of the brackets.
$x^2 - 6x + 2 = 0$	**A1**	Collected to zero: $x^2 - x - 2 - 5x + 4$.
$x = (6 \pm \sqrt{(36 - 8)}) / 2 = (6 \pm \sqrt{28}) / 2$	**M1**	Quadratic formula with a = 1, b = -6, c = 2. It does not factorise.
$x = 3 \pm \sqrt{7}$	**A1**	$\sqrt{28} = 2\sqrt{7}$, then divide both terms by 2.

The trap
Multiplying the fractions by their denominators but leaving the 1 on the right unmultiplied.

The habit that beats it
Write the common denominator on every term, including whole numbers, then clear it in one step.

Equations with algebraic fractions

In plain words

When x is in the bottom of a fraction, multiply every term by the common denominator. The equation then becomes an ordinary quadratic.

The formula or method

Common denominator = the product of the different denominators.
Quadratic formula: $x = (-b \pm \sqrt{(b^2 - 4ac)}) / 2a$
Values that make a denominator 0 are not allowed.

How to use it

1. Multiply every term, including whole numbers, by the common denominator.

2. Expand the brackets carefully and collect everything on one side.

3. Try to factorise. If it does not factorise, use the quadratic formula.

4. Simplify any surd, for example $\sqrt{28} = 2\sqrt{7}$, and check that no answer makes a denominator 0.

Worked mini-example
$2 / (x - 1) + 1 / (x + 1) = 1$ gives $2(x + 1) + (x - 1) = (x - 1)(x + 1)$, so $3x + 1 = x^2 - 1$ and $x^2 - 3x - 2 = 0$.

Spot it

A fraction equation that says 'give your answer in surd form' or 'in the form $a \pm \sqrt{b}$'.

Question 11

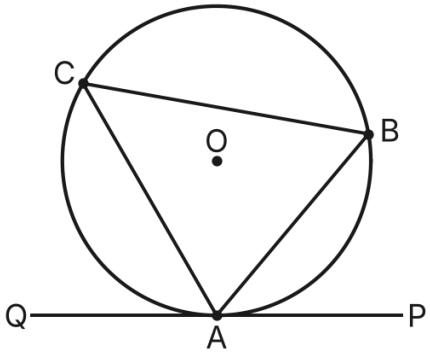
4 marks

Circles: Proof of a theorem

PAQ is the tangent at A to a circle, centre O. B and C are points on the circle. P and C are on opposite sides of AB.

Prove the alternate segment theorem: angle PAB = angle ACB. (4)

You may not quote the theorem. You are proving it.

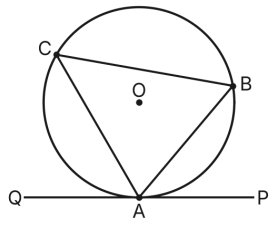


Working space

Question 11: marking and grade 9 answer

4 marks

PAQ is the tangent at A to a circle, centre O. B and C are points on the circle. P and C are on opposite sides of AB.
Prove the alternate segment theorem: angle PAB = angle ACB. (4)
You may not quote the theorem. You are proving it.



Line of the answer	Mark	How it is marked
Draw the diameter AD and join BD	**M1**	The construction that unlocks the proof. Without it you cannot start.
Angle PAD = 90° (tangent meets radius) and angle ABD = 90° (angle in a semicircle)	**C1**	Two reasons, both needed.
Angle PAB = 90° - angle BAD. In triangle ABD, angle ADB = 90° - angle BAD. So angle PAB = angle ADB	**M1**	Both are 90 minus the same angle.
Angle ADB = angle ACB (angles in the same segment), so angle PAB = angle ACB	**A1**	Links D to C and finishes the proof.

The trap
Quoting the alternate segment theorem as a reason. That is the thing being proved, so the answer scores zero.

The habit that beats it
Prove-the-theorem questions always allow the earlier theorems: tangent and radius, semicircle, same segment.

The alternate segment theorem

In plain words

The angle between a tangent and a chord equals the angle in the alternate segment, which is the segment on the other side of the chord.

The theorem

Theorem: the angle between a tangent and a chord at the point of contact equals the angle in the alternate segment.
Proof uses: tangent and radius at 90°, angle in a semicircle, and the angle sum of a triangle.

How to use it

1. Draw the diameter from A, through the centre, to a point D on the circle.

2. Angle DAP = 90° (tangent and radius). Angle ABD = 90° (angle in a semicircle).

3. Show PAB = 90° - BAD and ADB = 90° - BAD (angle sum of triangle ABD).

4. ADB = ACB (same segment), so PAB = ACB.

Worked mini-example
Tangent PAQ, angle PAB = 48°. The angle in the alternate segment, angle ACB, is also 48°.

Spot it

A tangent and a chord that meet at a point on the circle. Colour the angle and the segment on the far side.

Question 12

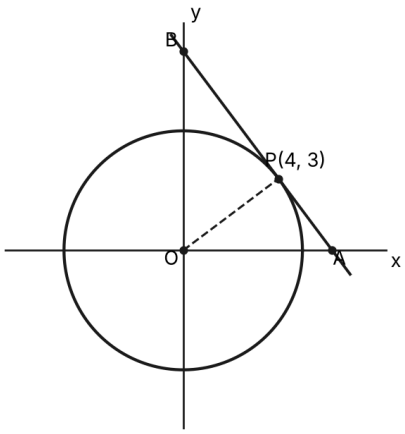
6 marks

Algebra: tangent to a circle

The circle $x^2 + y^2 = 25$ passes through P(4, 3). The tangent to the circle at P meets the x-axis at A and the y-axis at B.

(a) Find an equation of the tangent at P. Give it in the form $ax + by = c$. (3)

(b) Work out the area of triangle OAB. (3)

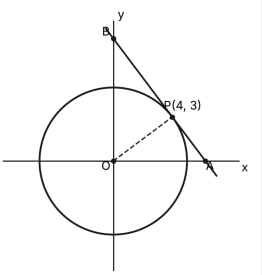


Working space

Question 12: marking and grade 9 answer

6 marks

The circle $x^2 + y^2 = 25$ passes through P(4, 3). The tangent to the circle at P meets the x-axis at A and the y-axis at B.
(a) Find an equation of the tangent at P. Give it in the form $ax + by = c$. (3)
(b) Work out the area of triangle OAB. (3)



Line of the answer	Mark	How it is marked
Gradient of OP = 3/4	**M1**	The radius from the centre O(0, 0) to P(4, 3).
Tangent is perpendicular to the radius, so its gradient is -4/3	**M1**	Negative reciprocal. Needs the reason in your head, and on paper.
$y - 3 = -(4/3)(x - 4)$, so $4x + 3y = 25$	**A1**	Multiply by 3 and rearrange to the required form.
A: $y = 0$ gives $x = 25/4$. B: $x = 0$ gives $y = 25/3$	**M1**	Intercepts from the equation of the tangent.
Area = ½ × 25/4 × 25/3	**M1**	Right-angled triangle with the axes, so the base and height are the intercepts.
= 625/24 square units	**A1**	Exact fraction. A rounded decimal is accepted only if the fraction is also shown.

The trap
Using the gradient of the radius (3/4) as the gradient of the tangent.

The habit that beats it
Radius first, then flip and negate. Check your tangent equation by substituting (4, 3).

Tangent to a circle at a point

In plain words

The tangent at a point on a circle is perpendicular to the radius there. Perpendicular lines have gradients that multiply to -1, which gives the equation of the tangent.

The formula or method

Circle centred at the origin: $x^2 + y^2 = r^2$.
Perpendicular gradients: $m_1 \times m_2 = -1$.
Line through (x_1, y_1): $y - y_1 = m(x - x_1)$.

How to use it

1. Find the gradient of the radius from the centre to the point.

2. Flip it and change the sign to get the gradient of the tangent.

3. Write the line through the point and rearrange to the form asked for.

4. For intercepts, put y = 0 or x = 0.

Worked mini-example
Circle $x^2 + y^2 = 13$ at (2, 3). Radius gradient 3/2, tangent gradient -2/3, so $y - 3 = -(2/3)(x - 2)$, which gives $2x + 3y = 13$.

Spot it

'Find an equation of the tangent at the point' on a circle centred at the origin.

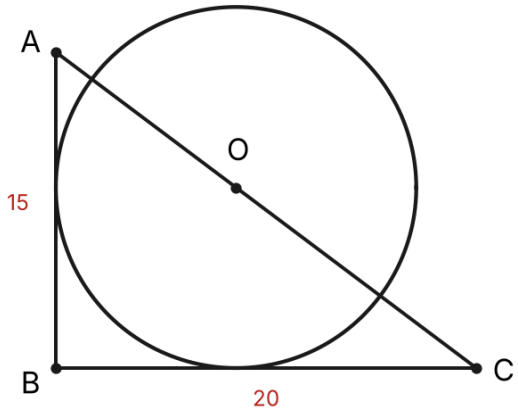
Question 13

6 marks

Circles: Show that

ABC is a right-angled triangle with angle ABC = 90°, AB = 15 cm and BC = 20 cm. A circle, centre O, has its centre on AC and touches AB and BC.

Show that the radius of the circle is 60 ÷ 7 cm. (6)

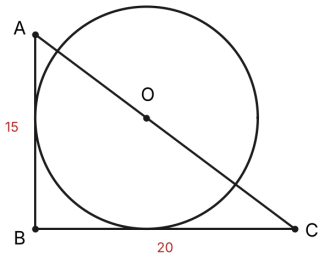


Working space

Question 13: marking and grade 9 answer

6 marks

ABC is a right-angled triangle with angle ABC = 90°, AB = 15 cm and BC = 20 cm. A circle, centre O, has its centre on AC and touches AB and BC. Show that the radius of the circle is 60 ÷ 7 cm. (6)



Line of the answer	Mark	How it is marked
The radii to the points of contact are perpendicular to AB and BC (tangent meets radius), so the perpendicular distance from O to each of AB and BC is r	**C1**	Reason needed. This makes r the height in two triangles.
Area of triangle ABC = ½ × 15 × 20 = 150	**M1**	Total area.
Join OB. Triangle ABO has base 15 and height r. Triangle BCO has base 20 and height r	**M1**	Splitting triangle ABC into two triangles at O.
O lies on AC, so the two areas add to 150: ½(15)r + ½(20)r = 150	**M1**	The key equation.
35r ÷ 2 = 150, so r = 300 ÷ 35	**M1**	Solving.
r = 60 ÷ 7 cm	**A1**	Cancel to the exact value. A decimal does not 'show that'.

The trap
Treating the circle as the incircle. It does not touch AC. Its centre lies ON AC.

The habit that beats it
Split the shape into triangles whose heights are the radius. Coordinates also work: centre (r, r), and r ÷ 20 + r ÷ 15 = 1.

Radius as height: splitting an area

In plain words

A circle that touches two sides of a triangle has a radius at 90° to each side, so the radius is the height of a small triangle on that side. Adding those small areas gives the whole area.

The theorem

Tangent and radius are perpendicular, so each radius is a height.
Area of the whole = sum of the areas of the smaller triangles from the centre.

How to use it

1. Mark the points of contact and the right angles with the radius r.

2. Find the total area of the big triangle from its two perpendicular sides.

3. Write each small triangle as ½ × side × r. Add them and set equal to the total.

4. Solve the linear equation for r.

Worked mini-example
Right-angled triangle with sides 6 and 8, and a circle touching both with its centre on the hypotenuse.
Area = 24 = ½ × 6 × r + ½ × 8 × r = 7r, so r = 24/7.

Spot it

A circle that touches the sides of a triangle, with its centre on a given line.

Question 14

6 marks

Surds: rectangle

A rectangle has width $(3 - \sqrt{2})$ cm and area 7 cm^2.

(a) Show that the length of the rectangle is $(3 + \sqrt{2})$ cm. (3)

(b) Show that the perimeter of the rectangle is 12 cm. (1)

(c) Work out the exact length of a diagonal of the rectangle. (2)

Working space

Question 14: marking and grade 9 answer

6 marks

A rectangle has width $(3 - \sqrt{2})$ cm and area 7 cm^2.
(a) Show that the length of the rectangle is $(3 + \sqrt{2})$ cm. (3)
(b) Show that the perimeter of the rectangle is 12 cm. (1)
(c) Work out the exact length of a diagonal of the rectangle. (2)

Line of the answer	Mark	How it is marked
Length = $7 / (3 - \sqrt{2})$	**M1**	Area divided by width.
$= 7(3 + \sqrt{2}) / (3 - \sqrt{2})(3 + \sqrt{2}) = 7(3 + \sqrt{2}) / (9 - 2)$	**M1**	Multiply top and bottom by the conjugate.
$= 7(3 + \sqrt{2}) / 7 = 3 + \sqrt{2}$	**A1**	The 7 cancels, which is the 'show that' finish.
Perimeter $= 2[(3 - \sqrt{2}) + (3 + \sqrt{2})] = 2 \times 6 = 12$ cm	**B1**	The surd terms cancel exactly.
Diagonal$^2 = (3 - \sqrt{2})^2 + (3 + \sqrt{2})^2 = (11 - 6\sqrt{2}) + (11 + 6\sqrt{2}) = 22$	**M1**	Pythagoras. Each square has the form $9 + 2 \pm 6\sqrt{2}$.
Diagonal $= \sqrt{22}$ cm	**A1**	Exact. The middle surd terms cancel.

The trap
Writing $(3 - \sqrt{2})^2 = 9 - 2$ and losing the middle term $-6\sqrt{2}$.

The habit that beats it
Expand every squared bracket with all three terms. In pairs like these the middle terms cancel, which tells you that you are right.

Working with surds

In plain words

A surd such as $\sqrt{2}$ is an exact value. Keeping answers in surd form is exact, and brackets with surds must be expanded in full.

The formula or method

$(a \pm b\sqrt{c})^2 = a^2 \pm 2ab\sqrt{c} + b^2c$
$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$
Area = length × width. Perimeter = 2(length + width).

How to use it

1. Expand squared brackets with three terms, and watch the sign of the middle term.

2. To divide by a surd, multiply top and bottom by its conjugate.

3. Add terms with the same surd and cancel terms of opposite sign.

4. For 'show that', write the exact value that the question gives as your last line.

Worked mini-example
$(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$, and $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1$.

Spot it

A shape with sides containing surds, and the words 'exact' or 'show that'.

Question 15

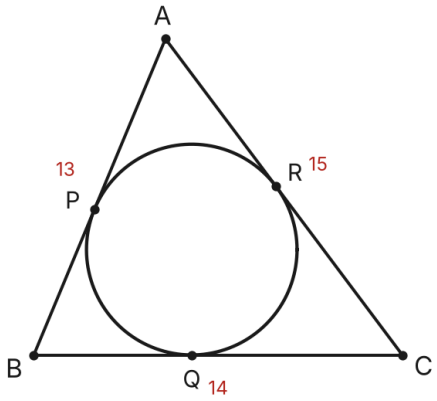
6 marks

Circles: Tangent lengths and the incircle

A circle touches the sides AB, BC and CA of triangle ABC at P, Q and R. AB = 13 cm, BC = 14 cm and CA = 15 cm.

(a) Work out the lengths AP, BQ and CR. (3)

(b) Work out the radius of the circle. (3)

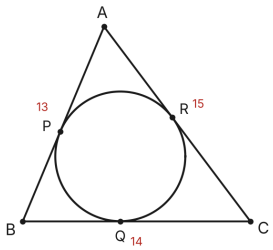


Working space

Question 15: marking and grade 9 answer

6 marks

A circle touches the sides AB, BC and CA of triangle ABC at P, Q and R. AB = 13 cm, BC = 14 cm and CA = 15 cm.
(a) Work out the lengths AP, BQ and CR. (3)
(b) Work out the radius of the circle. (3)



Line of the answer	Mark	How it is marked
(a) AP = AR, BP = BQ, CQ = CR (tangents from an external point are equal)	**C1**	The reason, written for all three vertices.
Let AP = x, BP = y, CQ = z. x + y = 13, y + z = 14, x + z = 15, so x + y + z = 21	**M1**	Three equations, add and halve.
AP = 21 - 14 = 7, BQ = 21 - 15 = 6, CR = 21 - 13 = 8	**A1**	All three correct.
(b) Drop the perpendicular AH to BC, with BH = d: $13^2 - d^2 = 15^2 - (14 - d)^2$, so d = 5, AH = 12. Area = ½ × 14 × 12 = 84	**M1**	The height needs algebra. A 13, 14, 15 triangle has height 12.
Area = ½ r (13 + 14 + 15), three triangles with the centre as apex and height r	**M1**	The incircle area formula, derived.
84 = 21r, so r = 4 cm	**A1**	Correct radius.

The trap
Using Heron's formula on a question that wants working. Also adding all three sides without halving (x + y + z = 21, not 42).

The habit that beats it
Whenever a circle touches a triangle's sides, define tangent-length variables immediately.

The incircle and tangent lengths

In plain words

A circle that touches all three sides of a triangle is its incircle. The two tangents from each vertex are equal, which turns the side lengths into simple equations.

The theorem

Tangents from a point are equal: AP = AR, BP = BQ, CQ = CR.
Tangent length from vertex A = s - a, where s is half the perimeter and a is the side opposite A.
Radius of the incircle = area ÷ s.

How to use it

1. Name the tangent lengths x, y and z at the three vertices.

2. Each side is the sum of two of them. Solve the three equations, or use s - a.

3. Find the area of the triangle (Heron's formula, or half base times height).

4. Radius = area ÷ s. Check by the tangent-radius right angle.

Worked mini-example
Triangle with sides 7, 8 and 9: s = 12. The tangent lengths from the vertices are 12 - 7 = 5, 12 - 8 = 4 and 12 - 9 = 3. They pair to give the sides: 5 + 4 = 9, 4 + 3 = 7, 3 + 5 = 8.

Spot it

A circle that touches all three sides of a triangle, with the three side lengths given.

Question 16

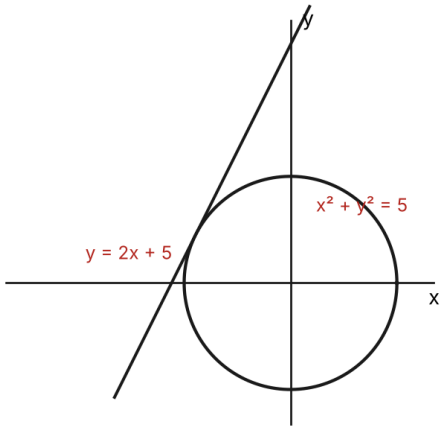
6 marks

Algebra: line and circle

The line $y = 2x + 5$ and the circle $x^2 + y^2 = 5$ are shown in the diagram.

(a) Show that the line is a tangent to the circle. (4)

(b) Find the coordinates of the point where the line touches the circle. (2)

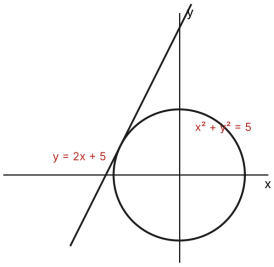


Working space

Question 16: marking and grade 9 answer

6 marks

The line $y = 2x + 5$ and the circle $x^2 + y^2 = 5$ are shown in the diagram.
(a) Show that the line is a tangent to the circle. (4)
(b) Find the coordinates of the point where the line touches the circle. (2)



Line of the answer	Mark	How it is marked
Substitute: $x^2 + (2x + 5)^2 = 5$	**M1**	Replace y in the circle equation.
$x^2 + 4x^2 + 20x + 25 = 5$, so $5x^2 + 20x + 20 = 0$	**A1**	$(2x + 5)^2$ expanded in full, then collected.
$x^2 + 4x + 4 = 0$, so $(x + 2)^2 = 0$	**M1**	Divide by 5 and factorise (or show the discriminant is 0).
One repeated root, so the line meets the circle at exactly one point: it is a tangent	**C1**	The conclusion must be stated in words.
$x = -2$	**B1**	The repeated root.
$y = 2(-2) + 5 = 1$, so the point is (-2, 1)	**A1**	Substitute into the line, not into the circle.

The trap
Expanding $(2x + 5)^2$ as $4x^2 + 25$, which forgets the 20x term.

The habit that beats it
Substitute the line into the circle, expand with three terms, and look at the discriminant. Zero means tangent.

A line meeting a circle

In plain words

To find where a line meets a circle, substitute the line into the circle. The resulting quadratic has two roots (two points), one repeated root (a tangent), or none (the line misses).

The formula or method

Substitute y = mx + c into the circle equation.
Discriminant > 0: two points. = 0: tangent. < 0: no intersection.
A repeated root gives the point of contact: x = -b / 2a.

How to use it

1. Substitute and expand the squared bracket with three terms.

2. Collect to a quadratic equal to zero and simplify by any common factor.

3. Test the discriminant, or factorise into a perfect square.

4. Put the repeated x back into the line to get y.

Worked mini-example
$y = x + k$ and $x^2 + y^2 = 2$: $2x^2 + 2kx + k^2 - 2 = 0$. The discriminant is $16 - 4k^2$, which is 0 for $k = \pm 2$.

Spot it

'Show that the line is a tangent' or 'find where the line meets the circle'.

Question 17

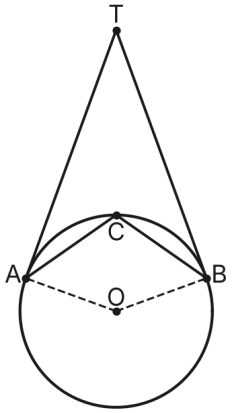
6 marks

Circles: Proof with algebra, reflex angle

TA and TB are tangents to a circle, centre O. C is a point on the **minor** arc AB.

(a) Prove that angle ACB = 90° + ½ angle ATB. (4)

(b) Angle ATB = $(3x - 20)°$ and angle ACB = $(5x + 10)°$. Work out x. (2)

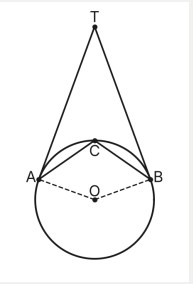


Working space

Question 17: marking and grade 9 answer

6 marks

TA and TB are tangents to a circle, centre O. C is a point on the **minor** arc AB.
(a) Prove that angle ACB = 90° + ½ angle ATB. (4)
(b) Angle ATB = (3*x* - 20)° and angle ACB = (5*x* + 10)°. Work out *x*. (2)



Line of the answer	Mark	How it is marked
(a) Angles OAT and OBT = 90° (tangent meets radius)	**C1**	Reason stated.
Angle AOB = 360 - 90 - 90 - angle ATB = 180 - angle ATB (angles in a quadrilateral)	**M1**	The non-reflex angle at the centre.
Reflex angle AOB = 360 - (180 - ATB) = 180 + ATB. C is on the minor arc, so angle ACB = ½ (180 + ATB) (angle at the centre is twice the angle at the circumference)	**C1**	The reflex angle is the centre angle for a point on the minor arc.
So angle ACB = 90° + ½ angle ATB	**A1**	Required result.
(b) 5*x* + 10 = 90 + ½(3*x* - 20)	**M1**	Using part (a).
10*x* + 20 = 180 + 3*x* - 20, so 7*x* = 140 and *x* = 20. (ATB = 40°, ACB = 110°)	**A1**	Correct value.

The trap
Using angle AOB as the centre angle. That gives the major-arc answer. C is on the MINOR arc, so the reflex angle is the centre angle.

The habit that beats it
Before using the centre theorem, say which arc the point sits on and which centre angle faces the other arc.

Angle at the centre, including the reflex angle

In plain words

The angle at the centre of a circle is twice the angle at the circumference when both stand on the same arc. If the point on the circumference is on the minor arc, the centre angle is the reflex angle.

The theorem

Theorem: angle at the centre = 2 × angle at the circumference (same arc).
Reflex angle AOB = 360° - angle AOB.
In kite OATB, angle AOB = 180° - angle ATB.

How to use it

1. Decide which arc the angle at the circumference stands on. Is C on the major or the minor arc?

2. If C is on the minor arc, the angle ACB stands on the major arc, so ACB = ½ × (reflex AOB).

3. Write reflex AOB = 360 - AOB and use the kite to link AOB to ATB.

4. Simplify, then use the result in an equation.

Worked mini-example
C on the minor arc and angle ACB = 110°. The reflex angle AOB = 220°, so angle AOB = 140°.

Spot it

'C is a point on the minor arc'. Expect an answer larger than 90°.

Question 18

6 marks

Surds: exact trigonometry

In triangle ABC, AB = $(2 + \sqrt{3})$ cm, AC = $(2 - \sqrt{3})$ cm and angle BAC = 60°.

(a) Show that the area of triangle ABC is $\sqrt{3} / 4$ cm^2. (2)

(b) Show that BC = $\sqrt{13}$ cm. (4)

Working space

Question 18: marking and grade 9 answer

6 marks

In triangle ABC, AB = (2 + √3) cm, AC = (2 - √3) cm and angle BAC = 60°.
(a) Show that the area of triangle ABC is √3 / 4 cm^2. (2)
(b) Show that BC = √13 cm. (4)

Line of the answer	Mark	How it is marked
Area = ½ × AB × AC × sin 60° = ½(2 + √3)(2 - √3) × √3/2	**M1**	Area formula with the included angle. sin 60° = √3 / 2.
(2 + √3)(2 - √3) = 4 - 3 = 1, so area = ½ × 1 × √3/2 = √3 / 4	**A1**	The product of conjugates is 1.
$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos 60°$	**M1**	Cosine rule with the angle between the two known sides.
$AB^2 = 7 + 4\sqrt{3}$ and $AC^2 = 7 - 4\sqrt{3}$, so $AB^2 + AC^2 = 14$	**M1**	Each square has three terms. The surd terms cancel.
2(AB)(AC) cos 60° = 2 × 1 × ½ = 1	**M1**	Uses AB × AC = 1 from part (a) and cos 60° = 1/2.
$BC^2 = 14 - 1 = 13$, so BC = √13 cm	**A1**	Show-that finish: state $BC^2 = 13$ before the root.

The trap
Using a calculator decimal for the answer to a 'show that' question. Exact values are the only proof.

The habit that beats it
Memorise sin, cos and tan of 30, 45 and 60 degrees. Look for conjugate pairs: their product is a whole number.

The cosine rule and exact trigonometry

In plain words

When two sides and the angle between them are known, the cosine rule gives the third side. The areas and sides can be exact if you know the trigonometric values of 30°, 45° and 60°.

The formula or method

Cosine rule: $a^2 = b^2 + c^2 - 2bc \cos A$.
Area = ½ bc sin A.
sin 60° = √3 / 2, cos 60° = 1/2, sin 30° = 1/2, cos 30° = √3 / 2, tan 45° = 1.

How to use it

1. Check that the angle is between the two known sides.

2. Substitute and keep every term exact. Square surds with three terms.

3. Look for conjugate pairs, since their product is a whole number.

4. Take the square root last and state it with its units.

Worked mini-example
Sides 2 and 3 with included angle 60°: $a^2 = 4 + 9 - 2 \times 2 \times 3 \times ½ = 7$, so $a = √7$.

Spot it

Surd side lengths and a special angle, and the word 'show that'.

Question 19

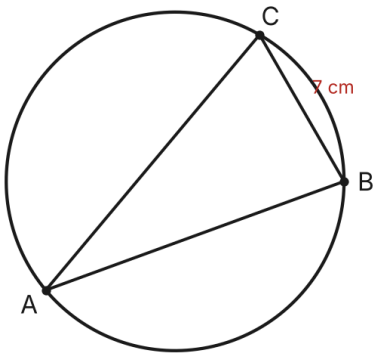
6 marks

Circles: Construction and sine rule

A, B and C are points on a circle. BC = 7 cm, angle BAC = 30° and angle ABC = 80°.

(a) Work out the radius of the circle. (3)

(b) Work out the length of AC. Give your answer correct to 3 significant figures. (3)

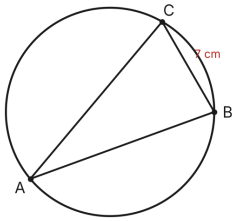


Working space

Question 19: marking and grade 9 answer

6 marks

A, B and C are points on a circle. BC = 7 cm, angle BAC = 30° and angle ABC = 80°.
(a) Work out the radius of the circle. (3)
(b) Work out the length of AC. Give your answer correct to 3 significant figures. (3)



Line of the answer	Mark	How it is marked
(a) Draw the diameter BD and join CD	**M1**	The unseen construction. It creates a right-angled triangle.
Angle BCD = 90° (angle in a semicircle) and angle BDC = angle BAC = 30° (angles in the same segment)	**C1**	Two reasons, one for each angle.
BD = 7 ÷ sin 30° = 14 cm, so the radius = 7 cm	**A1**	BD is the diameter.
(b) AC ÷ sin 80° = BC ÷ sin 30° (sine rule)	**M1**	Standard sine rule.
AC = 7 × sin 80° ÷ sin 30°	**M1**	Rearranged.
AC = **13.8 cm**	**A1**	Check: AC = 2r sin 80° = 14 sin 80°.

The trap
Treating BC as the diameter, or taking the radius as BC ÷ sin 30° (that gives the diameter, 14).

The habit that beats it
To link a chord to the radius, draw the diameter from one end of the chord. This is how 2R = a ÷ sin A is proved.

The sine rule and the circumscribed circle

In plain words

The sine rule links each side to the sine of its opposite angle, and the common ratio is the diameter of the circle through the three corners.

The theorem

a / sin A = b / sin B = c / sin C = 2R.
The angles in a triangle add to 180°.
Angle at the centre = 2 × angle at the circumference gives the same result.

How to use it

1. Match each side with its opposite angle.

2. Use a / sin A = 2R to find the radius of the circumscribed circle.

3. For the other side, use the third angle and the sine rule again.

4. Round only at the end, to the accuracy the question asks for.

Worked mini-example
BC = 10 and angle BAC = 30°: 2R = 10 / sin 30° = 20, so R = 10.

Spot it

Three points on a circle, one side and the angle opposite it.

Question 20

6 marks

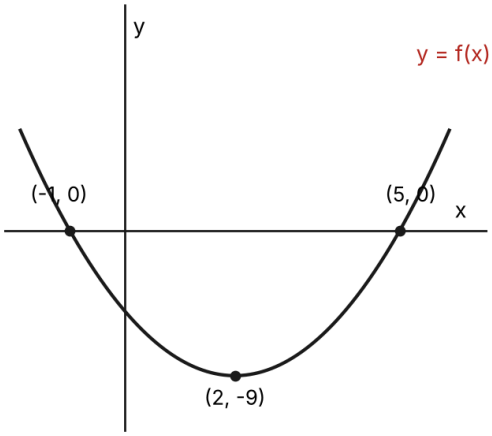
Algebra: graph transformations

The diagram shows the graph of $y = f(x)$ where $f(x) = x^2 - 4x - 5$. The minimum point is (2, -9) and the graph crosses the x-axis at (-1, 0) and (5, 0).

(a) Write down the coordinates of the minimum point of $y = f(x + 3)$. (2)

(b) Write down the roots of $f(2x) = 0$. (2)

(c) Write down the coordinates of the turning point of $y = 3 - f(x)$, and state whether it is a maximum or a minimum. (2)



Working space

Question 20: marking and grade 9 answer

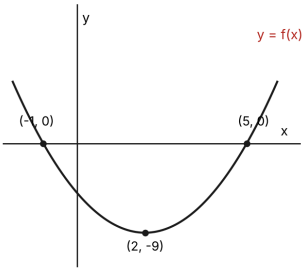
6 marks

The diagram shows the graph of $y = f(x)$ where $f(x) = x^2 - 4x - 5$. The minimum point is (2, -9) and the graph crosses the x-axis at (-1, 0) and (5, 0).

(a) Write down the coordinates of the minimum point of $y = f(x + 3)$. (2)

(b) Write down the roots of $f(2x) = 0$. (2)

(c) Write down the coordinates of the turning point of $y = 3 - f(x)$, and state whether it is a maximum or a minimum. (2)



Line of the answer	Mark	How it is marked
$f(x + 3)$ is a translation of 3 to the left: x-coordinate 2 - 3 = -1	**B1**	Inside the bracket, the shift is the opposite way to the sign.
The y-coordinate is unchanged: minimum at (-1, -9)	**B1**	A horizontal translation never changes y.
$f(2x)$ is a horizontal stretch with scale factor ½: divide the x-values by 2	**M1**	Inside the bracket: halve every x-value.
Roots $x = -1/2$ and $x = 5/2$	**A1**	Both roots. They are the old roots -1 and 5 halved.
$-f(x)$ reflects in the x-axis: (2, -9) becomes (2, 9). Then +3 moves it up: (2, 12)	**M1**	Two steps in the correct order: reflect, then translate.
(2, 12) and it is a maximum	**A1**	The reflection turns the minimum into a maximum.

The trap
Moving f(x + 3) to the right, or doubling the roots for f(2x).

The habit that beats it
Inside the bracket does the opposite and affects x. Outside the bracket does what it says and affects y.

Graph transformations

In plain words

Changes inside the bracket move the graph horizontally and do the opposite of what you expect. Changes outside the bracket move it vertically and do what they say.

The formula or method

f(x + a): translation of a units to the left. f(x) + a: up a units.
f(ax): horizontal stretch, scale factor 1/a. a f(x): vertical stretch, scale factor a.
-f(x): reflection in the x-axis. f(-x): reflection in the y-axis.

How to use it

1. Write down the original key points: turning point and where it crosses the axes.

2. Apply the change to the x-coordinates for anything inside the bracket.

3. Apply the change to the y-coordinates for anything outside the bracket.

4. When there are two changes, do the reflection or stretch first, then the translation.

Worked mini-example
f has a minimum at (2, -9). Then f(x - 1) has its minimum at (3, -9), 2f(x) has it at (2, -18), and f(x) + 5 has it at (2, -4).

Spot it

A diagram of y = f(x) with labelled points and a question about y = f(x + a) or y = -f(x).

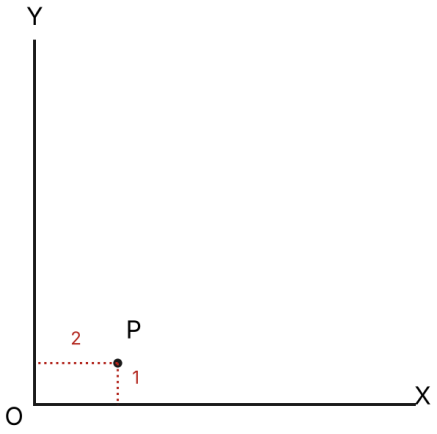
Question 21

6 marks

Circles: Two answers

OX and OY are perpendicular lines. P is a point 1 cm from OX and 2 cm from OY. A circle of radius *r* cm touches both OX and OY and passes through P.

Work out **all** the possible values of *r*. (6)

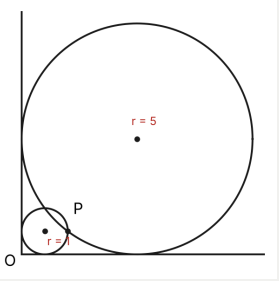


Working space

Question 21: marking and grade 9 answer

6 marks

OX and OY are perpendicular lines. P is a point 1 cm from OX and 2 cm from OY. A circle of radius r cm touches both OX and OY and passes through P.
Work out **all** the possible values of r. (6)



Line of the answer	Mark	How it is marked
The circle touches both lines, so its centre C is r from each (radii to the points of contact are perpendicular to the tangents). Take C = (r, r) with O at the origin	**C1**	Reason needed. This fixes where the centre is.
Horizontal distance from C to P = $\|r - 2\|$, vertical distance = $\|r - 1\|$, CP = r: $(r - 2)^2 + (r - 1)^2 = r^2$	**M1**	Pythagoras in the right-angled triangle from C to P.
$r^2 - 6r + 5 = 0$	**M1**	Expanding and collecting.
$(r - 1)(r - 5) = 0$	**M1**	Factorising.
$r = 5$	**A1**	First circle.
$r = 1$ (check: centre (1, 1) and P (2, 1) are 1 apart)	**A1**	Second circle. The small circle also fits. Both valid.

The trap
Finding r = 5 and drawing one circle. The quadratic has two roots and both give a real circle.

The habit that beats it
Test every root against the diagram before rejecting any. A 'possible' or 'all' question almost always has a second answer.

Circles that touch two lines

In plain words

A circle that touches two perpendicular lines has its centre at the same distance from both lines. The distance equals the radius, so the centre is at (r, r). Any point on the circle is then at distance r from it.

The theorem

Centre (r, r) when the circle touches both axes in the first quadrant.
Circle equation: $(x - r)^2 + (y - r)^2 = r^2$.
A point on the circle must satisfy it.

How to use it

1. Place the axes along the two lines and find the coordinates of P.

2. Write the centre as (r, r) and substitute P into the circle equation.

3. Expand and collect to a quadratic in r. Solve it.

4. Check that both values are possible, and draw both circles.

Worked mini-example
A circle touching both axes passes through (4, 2): $(4 - r)^2 + (2 - r)^2 = r^2$, so $r^2 - 12r + 20 = 0$ and r = 2 or r = 10.

Spot it

The word 'all' in 'all the possible values of r'. Expect two answers.

Question 22

6 marks

Algebra: composite functions

f(x) = 3x - 2 and g(x) = x^2 + 4

(a) Show that gf(x) = 9x^2 - 12x + 8. (2)

(b) Solve gf(x) = fg(x). Give your solutions in surd form. (4)

Working space

Question 22: marking and grade 9 answer

6 marks

f(x) = 3x - 2 and g(x) = x^2 + 4
(a) Show that gf(x) = 9x^2 - 12x + 8. (2)
(b) Solve gf(x) = fg(x). Give your solutions in surd form. (4)

Line of the answer	Mark	How it is marked
gf(x) = g(3x - 2) = (3x - 2)^2 + 4	**M1**	f is applied first, then g.
= 9x^2 - 12x + 4 + 4 = 9x^2 - 12x + 8	**A1**	Expand the bracket with three terms.
fg(x) = 3(x^2 + 4) - 2 = 3x^2 + 10	**M1**	g first, then f.
9x^2 - 12x + 8 = 3x^2 + 10, so 6x^2 - 12x - 2 = 0, so 3x^2 - 6x - 1 = 0	**M1**	Collect to zero and divide by 2.
x = (6 ± √(36 + 12)) / 6 = (6 ± √48) / 6	**M1**	Quadratic formula. Note the +12 from -4 × 3 × -1.
x = 1 ± 2√3 / 3	**A1**	√48 = 4√3, then divide each term by 6.

The trap
Doing the functions in the wrong order. gf means f first. Also, a sign slip in the discriminant when c is negative.

The habit that beats it
Read gf(x) from the inside out: the function nearest to x acts first.

Composite functions

In plain words

A composite function applies one function to the output of another. The function written closest to x acts first.

The formula or method

fg(x) = f(g(x)): g first, then f.
gf(x) = g(f(x)): f first, then g.
In general fg(x) and gf(x) are different.

How to use it

1. Write the inner function in brackets in place of x in the outer function.

2. Expand brackets fully. A squared bracket has three terms.

3. To solve an equation such as gf(x) = fg(x), set the two expressions equal and collect to zero.

4. Use the quadratic formula for surd answers.

Worked mini-example
f(x) = x + 1 and g(x) = 2x. fg(x) = 2x + 1 and gf(x) = 2(x + 1) = 2x + 2.

Spot it

Two functions and the notation fg(x) or gf(x).

Question 23

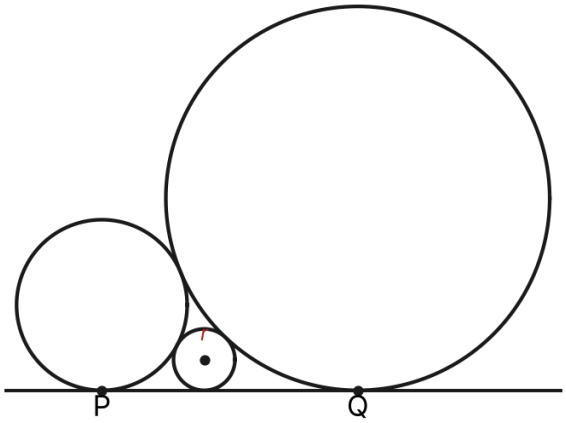
6 marks

Circles: Touching circles and a gap circle

Circles of radii 4 cm and 9 cm touch each other. Both touch a straight line, at P and Q. A third circle of radius *r* cm lies in the gap and touches both circles and the straight line.

(a) Show that PQ = 12 cm. (2)

(b) Work out the value of *r*. (4)

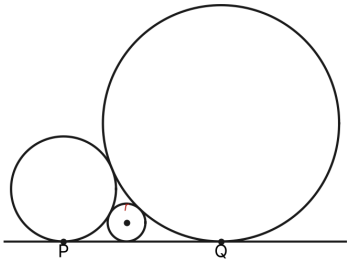


Working space

Question 23: marking and grade 9 answer

6 marks

Circles of radii 4 cm and 9 cm touch each other. Both touch a straight line, at P and Q. A third circle of radius r cm lies in the gap and touches both circles and the straight line.
(a) Show that PQ = 12 cm. (2)
(b) Work out the value of r. (4)



Line of the answer	Mark	How it is marked
(a) Distance between the centres = 4 + 9 = 13. The radii to P and Q are perpendicular to the line, so the right-angled triangle has hypotenuse 13 and one side 9 - 4 = 5	**M1**	The construction. Subtract the radii, do not add.
$PQ^2 = 13^2 - 5^2 = 144$, so PQ = 12 cm	**A1**	Show it fully.
(b) Let the third circle touch the line at R. For the 4 cm circle: $PR^2 = (4 + r)^2 - (4 - r)^2 = 16r$, so $PR = 4\sqrt{r}$	**M1**	Same right-angled-triangle idea, with the centres 4 + r apart.
For the 9 cm circle: $RQ^2 = (9 + r)^2 - (9 - r)^2 = 36r$, so $RQ = 6\sqrt{r}$	**M1**	Repeat.
PR + RQ = PQ: $10\sqrt{r} = 12$	**M1**	Link the two lengths through part (a).
$\sqrt{r} = 1.2$, so $r = 1.44$ cm	**A1**	Square 1.2.

The trap
Not using part (a) in part (b). The two parts are one chain.

The habit that beats it
With touching circles, draw the line of centres and the radii to the contact points, then look for a right-angled triangle.

Touching circles and the line of centres

In plain words

When two circles touch, the point of contact lies on the line joining the centres, and the distance between the centres is the sum of the radii. Two circles on a straight line give a right-angled triangle.

The theorem

Distance between centres = $r_1 + r_2$.
Horizontal gap between two circles on a line: $\sqrt{((r_1 + r_2)^2 - (r_1 - r_2)^2)} = 2\sqrt{(r_1 r_2)}$.
Gap circle: $1 / \sqrt{r} = 1 / \sqrt{r_1} + 1 / \sqrt{r_2}$.

How to use it

1. Draw the centres and join them. Drop a perpendicular to make a right-angled triangle.

2. The hypotenuse is the sum of the radii. The vertical side is the difference of the radii.

3. Use Pythagoras for the horizontal gap between the contact points.

4. For the third circle, add the two gaps and equate with the whole gap.

Worked mini-example
Circles of radius 1 and 4 touch. Centres are 5 apart and 3 apart vertically, so the gap along the line = $\sqrt{(25 - 9)} = 4 = 2\sqrt{(1 \times 4)}$.

Spot it

Circles touching each other and a straight line, with a gap between them.

Question 24

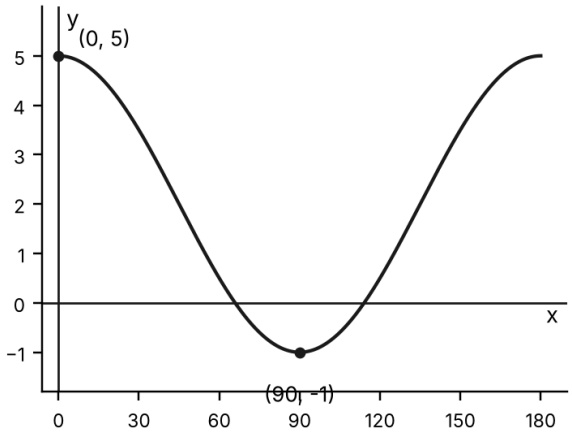
6 marks

Algebra: trigonometric graphs

The diagram shows part of the graph of $y = a\cos(bx°) + c$ for $0 \leq x \leq 180$. The maximum point is (0, 5) and the minimum point is (90, -1).

(a) Find the values of a, b and c. (3)

(b) Hence solve $a\cos(bx°) + c = 0.5$ for $0 \leq x \leq 180$. (3)

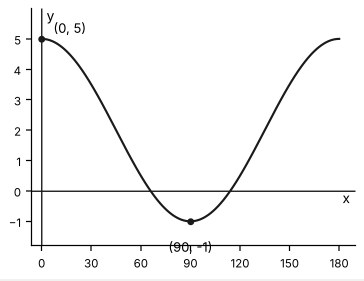


Working space

Question 24: marking and grade 9 answer

6 marks

The diagram shows part of the graph of $y = a\cos(bx°) + c$ for $0 \leq x \leq 180$.
The maximum point is (0, 5) and the minimum point is (90, -1).
(a) Find the values of a, b and c. (3)
(b) Hence solve $a\cos(bx°) + c = 0.5$ for $0 \leq x \leq 180$. (3)



Line of the answer	Mark	How it is marked
$c = (5 + (-1)) / 2 = 2$	**B1**	The midline is halfway between the maximum and the minimum.
$a = (5 - (-1)) / 2 = 3$	**B1**	The amplitude is half the distance between the maximum and the minimum.
Maximum at 0, next minimum at 90, so one full cycle is 180°: $b = 360 / 180 = 2$	**B1**	Half a cycle is 90°, so the period is 180°.
$3\cos(2x°) + 2 = 0.5$, so $\cos(2x°) = -0.5$	**M1**	Isolate the cosine.
$2x = 120$ or 240	**M1**	Inverse cosine gives 120°. The second solution is 360 - 120 = 240°.
$x = 60$ and $x = 120$	**A1**	Halve both. A range of 0 to 180 for x means 0 to 360 for 2x.

The trap
Dividing by b only for the first solution, or using a period of 90° instead of 180°.

The habit that beats it
Read the maximum and minimum first for a and c. Then the distance from a maximum to the next minimum is half a period.

Trigonometric graphs

In plain words

A graph of y = a cos(bx) + c is a stretched and shifted cosine wave. The numbers a, b and c tell you its height, its width and its middle line.

The formula or method

Amplitude a = (maximum - minimum) / 2.
Midline c = (maximum + minimum) / 2.
Period = 360° / b.

How to use it

1. Read the maximum and minimum from the graph. Calculate a and c.

2. Measure the distance between a maximum and the next maximum, or double the distance from a maximum to a minimum, for the period. Then b = 360 / period.

3. To solve an equation, isolate the cosine first.

4. Find the first solution with the inverse cosine, then 360° minus it, and divide by b.

Worked mini-example
y = 2 sin(3x) + 1: maximum 3, minimum -1, period 360 / 3 = 120°.

Spot it

A labelled graph and 'find the values of a, b and c', followed by 'hence solve'.

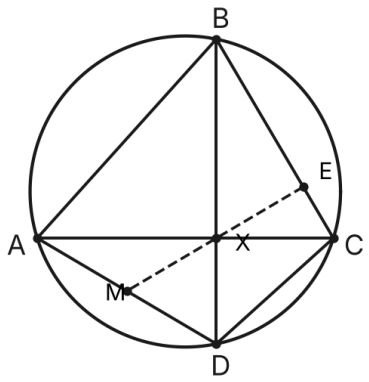
Question 25

6 marks

Circles: Brahmagupta's theorem

ABCD is a cyclic quadrilateral. The diagonals AC and BD meet at X at right angles. The line through X perpendicular to BC meets BC at E and AD at M.

Prove that M is the midpoint of AD. (6)

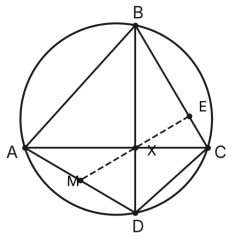


Working space

Question 25: marking and grade 9 answer

6 marks

ABCD is a cyclic quadrilateral. The diagonals AC and BD meet at X at right angles. The line through X perpendicular to BC meets BC at E and AD at M.
Prove that M is the midpoint of AD. (6)



Line of the answer	Mark	How it is marked
Let angle DBC = θ. Triangle BXE is right-angled at E, so angle BXE = 90° - θ	**M1**	Start from a named angle. Triangle BXE has a right angle at E.
Angle MXD = angle BXE = 90° - θ (vertically opposite angles)	**C1**	Reason needed.
Triangle BXC is right-angled at X, so angle ACB = 90° - θ	**M1**	The right angle at X is the diagonal condition.
Angle ADB = angle ACB = 90° - θ (angles in the same segment). So angle MDX = angle MXD, triangle MXD is isosceles and MX = MD	**C1**	Same segment, then isosceles.
Angle MXA = 90° - angle MXD = θ (angle AXD = 90°). Angle MAX = angle DAC = angle DBC = θ (same segment). So MX = MA	**C1**	The second half. Most students never find it.
MA = MX = MD, so M is the midpoint of AD	**A1**	Conclusion.

The trap
Assuming M is the midpoint (the thing to prove), or proving only MX = MD and stopping. You need both halves.

The habit that beats it
Give the angle you start from a name (θ), then express everything in terms of it. Right angles at X turn every angle into 90 minus something.

Proving that a point is a midpoint

In plain words

To show M is the midpoint of AD, show that MA = MX and MX = MD. Each equality comes from an isosceles triangle, which comes from equal angles.

The theorem

Angles in the same segment are equal.
Vertically opposite angles are equal.
Equal angles in a triangle mean equal sides opposite them.

How to use it

1. Give one angle a name, θ, and write other angles as 90° - θ using the right angles at X.

2. Use same-segment angles to carry θ around the circle.

3. Show two triangles are isosceles and state the two equal lengths.

4. Conclude: MA = MX = MD, so M is the midpoint. State exactly what you were asked to prove.

Worked mini-example
If MX = MA and MX = MD, then MA = MD. Show both halves. Proving only one is worth no more than half the marks.

Spot it

Perpendicular diagonals of a cyclic quadrilateral, with a line from the crossing point to a side.

Score tracker

Type	Questions	Marks available	Your score
Circles	1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25	66	
Algebra	2, 6, 10, 12, 16, 20, 22, 24	44	
Surds	4, 8, 14, 18	20	
Total		**130**	

Reading your score

Tally the lost marks by code: reasons (C), method (M) or accuracy (A), and fix the biggest group first. Also check which traps caught you. If the same one caught you twice, that is your next topic.